INTEGRAL CALCULUS (MATH 106)

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Weekly Objectives

Week 11: Area between curves and Volume of a solid revolution.

The student is expected to be able to:

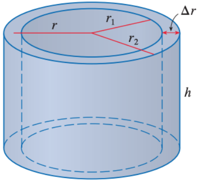
1. Calculate the volume of a solid revolution using the Cylindrical shells method.
2. Calculate the arc length.
3. Calculate the area of a surface of revolution.

Volume Of A Solid Revolution (Cylindrical shells method)

The method of cylindrical shells

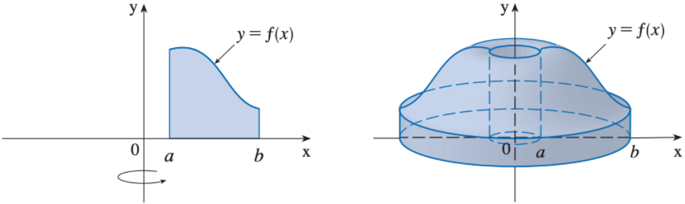
the cylindrical shell with inner radius r_1, outer radius r_2 , and height h. Its volume V is calculated by subtracting the volume V_1 of the inner cylinder from the volume V_2 of the outer cylinder:

$$V = V_2 - V_1 = \pi r_2^2 h - \pi r_1^2 h$$
$$= \pi(r_2^2 - r_1^2)h = \pi(r_2 - r_1)(r_2 + r_1)h$$
$$= 2\pi \frac{r_2 + r_1}{2} h(r_2 - r_1) \Rightarrow V = 2\pi r h \Delta r$$



Volume Of A Solid Revolution (Cylindrical shells method)

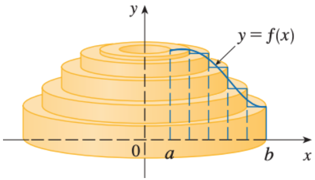
let be the solid obtained by rotating about the -axis the region bounded by $y = f(x)$,
where $f(x) \geq 0$, $y = 0$, $x = a$ and $x = b$, where $b > a \geq 0$.

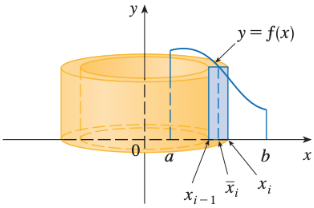


Volume Of A Solid Revolution (Cylindrical shells method)

We divide the interval into n subintervals $[x_{i-1}, x_{i+1}]$ of equal width and let $\overline{x_i}$ be the midpoint of the i th subinterval. If the rectangle with base $[x_{i-1}, x_i]$ and height $f(\overline{x}_i)$ is rotated about the $y-$ axis then the result is a cylindrical shell with average radius $\overline{x}_i$ height $f(\overline{x}_i)$ and thickness Δx so its volume is:

$$V_i = (2\pi)\overline{x}_i[f(\overline{x}_i)]\Delta x$$





Volume Of A Solid Revolution (Cylindrical shells method)

An approximation to the volume of is given by the sum of the volumes of these shells:

$$V \approx \sum_{i=1}^{n} V_i = \sum_{i=1}^{n} 2\pi \overline{x}_i [f(\overline{x}_i)] \Delta x$$

This approximation appears to become better as $n \to \infty$ But, from the definition of an integral, we know that

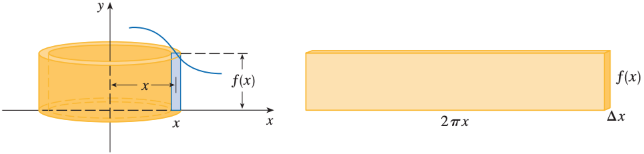
$$\lim_{n\to\infty} \sum_{i=1}^{n} 2\pi \overline{x}_i [f(\overline{x}_i)] \Delta x = \int_a^b 2\pi x f(x)\, dx$$

The volume of the solid, obtained by rotating about the $y-$axis the region under the curve $y = f(x)$ from a to b, is

$$V = \int_a^b 2\pi x f(x) dx \qquad where \quad 0 \leq a < b$$

The best way to remember the last Formula is to think of a typical shell, cut and flattened as in Figure with radius x, circumference $2\pi x$, height $f(x)$ and thickness Δx or dx :

$$\int_a^b \underbrace{(2\pi x)}_{\text{circumference}} \underbrace{[f(x)]}_{\text{height}} dx$$

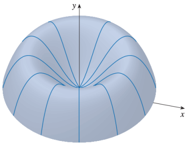


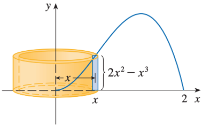
Example 2.1

Find the volume of the solid obtained by rotating about the $y-$axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$

by the shell method, the volume is

$$V = \int_0^2 (2\pi x)(2x^2 - x^3)\, dx = 2\pi \int_0^2 (2x^3 - x^4)\, dx = 2\pi \left[\frac{x^4}{2} - \frac{x^5}{5}\right]_0^2$$
$$= 2\pi(8 - \tfrac{32}{5}) = \tfrac{16}{5}\pi$$





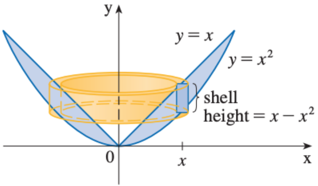
Example 2.2

Find the volume of the solid obtained by rotating about the y−axis the region between $y = x$ and $y = x^2$.

$$V = \int_0^1 (2\pi x)(x - x^2)\ dx$$

$$= 2\pi \int_0^1 (x^2 - x^3)\ dx$$

$$= 2\pi \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{\pi}{6}$$



Example 2.3

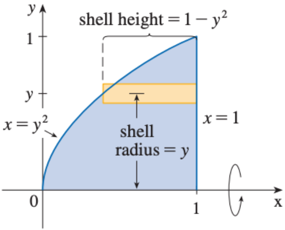
Use cylindrical shells to find the volume of the solid obtained by rotating about the $x-$axis the region under the curve $y = \sqrt{x}$ from 0 to 1.

For rotation about the $x-$axis we see that a typical shell has radius y, circumference $2\pi y$, and height $1 - y^2$. So the volume is

$$V = \int_0^1 (2\pi y)(1 - y^2)\ dy$$

$$= 2\pi \int_0^1 (y - y^3)\ dy$$

$$= 2\pi \left[\frac{y^2}{2} - \frac{y^4}{4}\right]_0^1 = \frac{\pi}{2}$$

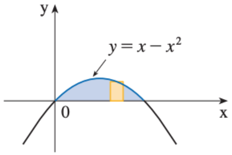


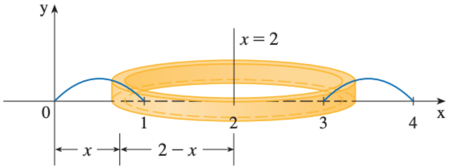
Example 2.4

Find the volume of the solid obtained by rotating the region bounded by $y = x - x^2$ and $y = 0$ about the line $x = 2$.

the region and a cylindrical shell formed by rotation about the line $x = 2$. It has radius $2 - x$, circumference $2\pi(2 - x)$, and height $x - x^2$.

$$V = \int_0^1 2\pi(2 - x)(x - x^2)\, dx = 2\pi \int_0^1 (x^3 - 3x^2 + 2x)\, dx$$
$$= 2\pi\left[\frac{x^4}{4} - x^3 + x^2\right]_0^1 = \frac{\pi}{2}$$





Arc Length

Definition 3.1

1. *If $f(x)$ is continuous function on the interval $[a, b]$, then the arc length of $f(x)$ from $x = a$ to $x = b$ is:*

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

2. *If $g(y)$ is continuous function on the interval $[c, d]$, then the arc length of $g(y)$ from $y = c$ to $y = d$ is:*

$$L = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy$$

Arc Length (Example)

Example 3.1

Determine the length of $y = \ln(\sec x)$ *between* $0 \leq x \leq \frac{\pi}{4}$

$f'(x) = \frac{\sec x \tan x}{\sec x} = \tan x \Rightarrow [f'(x)]^2 = \tan^2 x$

$\sqrt{1 + [f'(x)]^2} = \sqrt{1 + \tan^2 x} = \sqrt{\sec^2 x} = |\sec x| = \sec x$

The arc length is then,

$$\int_0^{\frac{\pi}{4}} \sec x \; dx = \Big[\ln|\sec x + \tan x| \Big]_0^{\frac{\pi}{4}} = \ln(\sqrt{2} + 1)$$

Arc Length (Example)

Example 3.2

Determine the length of $x = \frac{2}{3}(y-1)^{\frac{3}{2}}$ *between* $1 \le y \le 4$

$\frac{dx}{dy} = (y-1)^{\frac{1}{2}} \Rightarrow \sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1+y-1} = \sqrt{y}$

The arc length is then,

$$L = \int_1^4 \sqrt{y}\, dy = \frac{2}{3} y^{\frac{3}{2}} \Big|_1^4 = \frac{14}{3}$$

Arc Length (Example)

Example 3.3

Determine the length of $x = \frac{1}{2}y^2$ between $0 \leq x \leq \frac{1}{2}$. Assume that y is positive.

$\frac{dx}{dy} = y \quad \Rightarrow \quad \sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1 + y^2}$

Before writing down the length notice that we were given x limits and we will need y limits. $0 \leq y \leq 1$

The integral for the arc length is then,

$$L = \int_0^1 \sqrt{1 + y^2}\, dy$$

Arc Length (Example)

$$L = \int_0^1 \sqrt{1+y^2}\,dy$$

This integral will require the following trig substitution.

$y = \tan\theta \qquad dy = \sec^2\theta\, d\theta$

$$y = 0 \quad \Rightarrow \quad 0 = \tan\theta \quad \Rightarrow \quad \theta = 0$$

$$y = 1 \quad \Rightarrow \quad 1 = \tan\theta \quad \Rightarrow \quad \theta = \frac{\pi}{4}$$

$$\sqrt{1+y^2} = \sqrt{1+\tan^2\theta} = \sqrt{\sec^2\theta} = |\sec\theta| = \sec\theta$$

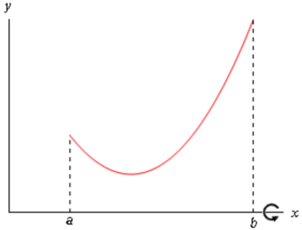
Arc Length (Example)

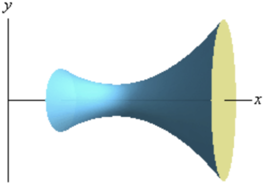
The length is then,

$$\begin{aligned} L &= \int_0^{\frac{\pi}{4}} \sec^3\theta \, d\theta \\ &= \frac{1}{2}\left(\sec\theta\tan\theta + \ln\left|\sec\theta + \tan\theta\right|\right)\Big|_0^{\frac{\pi}{4}} \\ &= \frac{1}{2}\left(\sqrt{2} + \ln\left(1+\sqrt{2}\right)\right) \end{aligned}$$

Area of a Surface of Revolution

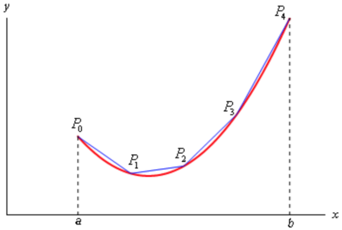
Let $f(x)$ be a nonnegative smooth function over the interval $[a, b]$. We wish to find the surface area of the surface of revolution created by revolving the graph of $y = f(x)$ around the $x-$axis as shown in the following figure.





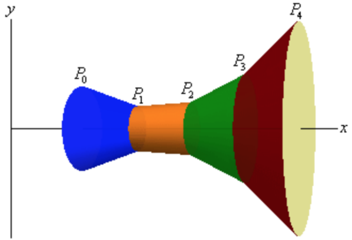
Area of a Surface of Revolution

1. We'll start by dividing the interval into n equal subintervals of width Δx
2. On each subinterval we will approximate the function with a straight line that agrees with the function at the endpoints of each interval.
3. Here is a sketch of that for our representative function using $n = 4$



Area of a Surface of Revolution

Now, rotate the approximations about the $x-$axis and we get the following solid.



The approximation on each interval gives a distinct portion of the solid and to make this clear each portion is colored differently.

Area of a Surface of Revolution

The area of each of these is:

$$A = 2\pi rl$$

where,

$$r = \frac{1}{2}(r_1 + r_2) \qquad \begin{aligned} r_1 &= \text{radius of right end} \\ r_2 &= \text{radius of left end} \end{aligned}$$

and l is the length of the slant of each interval.

Area of a Surface of Revolution

We know from the previous section that,

$$|P_{i-1}\ P_i| = \sqrt{1 + [f'(x_i^*)]^2}\ \Delta x \quad \text{where } x_i^* \text{ is some point in } [x_{i-1}, x_i]$$

Before writing down the formula for the surface area we are going to assume that Δx is "small" and since $f(x)$ is continuous we can then assume that,

$$f(x_i) \approx f(x_i^*) \qquad \text{and} \qquad f(x_{i-1}) \approx f(x_i^*)$$

Area of a Surface of Revolution

So, the surface area of each interval $[x_{i-1}, x_i]$ is approximately,

$$A_i = 2\pi \left(\frac{f(x_i) + f(x_{i-1})}{2} \right) |P_{i-1}\ P_i|$$
$$\approx 2\pi f(x_i^*) \sqrt{1 + [f'(x_i^*)]^2}\ \Delta x$$

The surface area of the whole solid is then approximately,

$$S \approx \sum_{i=1}^{n} 2\pi f(x_i^*) \sqrt{1 + [f'(x_i^*)]^2}\ \Delta x$$

Area of a Surface of Revolution

and we can get the exact surface area by taking the limit as n goes to infinity.

$$\begin{aligned} S &= 2\pi \lim_{n\to\infty} \sum_{i=1}^{n} f\left(x_i^*\right)\sqrt{1+\left[f'\left(x_i^*\right)\right]^2}\ \Delta x \\ &= 2\pi \int_a^b f\left(x\right)\sqrt{1+\left[f'\left(x\right)\right]^2}\, dx \end{aligned}$$

If we wanted to we could also derive a similar formula for rotating $x = h\left(y\right)$ on $[c, d]$ about the $y-$axis. This would give the following formula.

$$S = 2\pi \int_c^d h\left(y\right)\sqrt{1+\left[h'\left(y\right)\right]^2}\, dy$$

Area of a Surface of Revolution (Example)

Example 4.1

Determine the surface area of the solid obtained by rotating $y = \sqrt{9 - x^2}, -2 \leq x \leq 2$ *about the* $x-$*axis.*

$$S = 2\pi \int_a^b f(x) \sqrt{1 + [f'(x)]^2}\, dy$$

$$\frac{dy}{dx} = \frac{1}{2}\left(9 - x^2\right)^{-\frac{1}{2}}(-2x) = -\frac{x}{(9 - x^2)^{\frac{1}{2}}}$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{x^2}{9 - x^2}} = \sqrt{\frac{9}{9 - x^2}} = \frac{3}{\sqrt{9 - x^2}}$$

Area of a Surface of Revolution (Example)

Here's the integral for the surface area,

$$S = 2\pi \int_{-2}^{2} f(x) \frac{3}{\sqrt{9 - x^2}}\, dx$$

$$S = 2\pi \int_{-2}^{2} \sqrt{9 - x^2} \frac{3}{\sqrt{9 - x^2}}\, dx$$

$$= 6\pi \int_{-2}^{2} dx = 24\pi$$

Area of a Surface of Revolution (Example)

Example 4.2

Determine the surface area of the solid obtained by rotating $y = \sqrt[3]{x}, 1 \leq y \leq 2$ about the y−axis.

Solution

$$S = 2\pi \int_c^d h(y) \sqrt{1 + [h'(y)]^2}\, dy$$

$$x = h(y) = y^3 \qquad \frac{dx}{dy} = 3y^2$$

$$\sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1 + 9y^4}$$

The surface area is then,

$$S = 2\pi \int_1^2 h(y)\sqrt{1+9y^4}\,dy$$

$$S = 2\pi \int_1^2 y^3\sqrt{1+9y^4}\,dy \qquad u = 1+9y^4$$

$$= \frac{\pi}{18}\int_{10}^{145} \sqrt{u}\,du$$

$$= \frac{\pi}{27}\left(145^{\frac{3}{2}} - 10^{\frac{3}{2}}\right) = 199.48$$