INTEGRAL CALCULUS (MATH 106)

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Weekly Objectives

Week 12: Arc length and surface area of a parametric equation, and polar coordinates

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The student is expected to be able to:

1. Know the definition of parametric equations
2. Calculate the slope of the tangent line to parametric curve.
3. Calculate arc length of a parametric equations.
4. Calculate the surface area generated by revolving a parametric curve.

Parametric equations

To this point we've looked almost exclusively at functions in the form $y = f(x)$ or $x = h(y)$
It is easy to write down the equation of a circle centered at the origin with radius r.

$$x^2 + y^2 = r^2$$

However, we will never be able to write the equation of a circle down as a single equation in either of the forms above. Sure we can solve for x or y as the following two formulas show

$$y = \pm\sqrt{r^2 - x^2} \qquad x = \pm\sqrt{r^2 - y^2}$$

but there are in fact two functions in each of these. Each formula gives a portion of the circle.

Parametric equations

$$y = \sqrt{r^2 - x^2} \qquad \text{(top)} \qquad x = \sqrt{r^2 - y^2} \qquad \text{(right side)}$$

$$y = -\sqrt{r^2 - x^2} \qquad \text{(bottom)} \qquad x = -\sqrt{r^2 - y^2} \qquad \text{(left side)}$$

There are also a great many curves out there that we can't even write down as a single equation in terms of only x and y. So, to deal with some of these problems we introduce **parametric equations.**

Parametric equations

Instead of defining y in terms of x, $y = f(x)$ or x in terms of y $x = h(y)$ we define both x and y in terms of a third variable called a parameter as follows,

$$x = f(t) \qquad y = g(t)$$

This third variable is usually denoted by t.
Each value of t defines a point $(x, y) = (f(t), g(t))$ that we can plot. The collection of points that we get by letting t be all possible values is the graph of the parametric equations and is called the **parametric curve.**

Parametric equations (Example)

Example 2.1

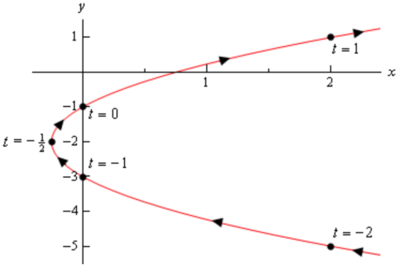
Sketch the parametric curve for the following set of parametric equations.

$$x = t^2 + t \qquad y = 2t - 1 \quad -2 \leq t \leq 2$$

At this point our only option for sketching a parametric curve is to pick values of t, plug them into the parametric equations and then plot the points. So, let's plug in some t's.

Parametric equations (Example)

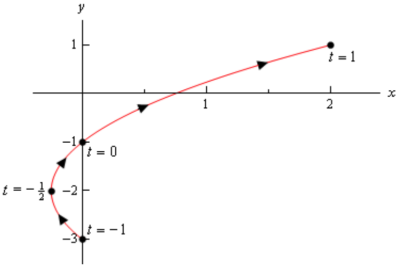
t	x	y
-2	2	-5
-1	0	-3
$-\frac{1}{2}$	$-\frac{1}{4}$	-2
0	0	-1
1	2	1



Example 2.2

Sketch the parametric curve for the following set of parametric equations.

$$x = t^2 + t \qquad y = 2t - 1 \qquad -1 \leq t \leq 1$$



The slope of the tangent line to a parametric curve

If $C : x = x(t), y = y(t); a \leq t \leq b$ is a differentiable parametric curve then the slope of the tangent line to C at $t_0 \in [a, b]$ is:

$$m = \frac{dy}{dx}\Big|_{t=t_0} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}\Big|_{t=t_0}$$

Remark

1. The tangent line to the parametric curve is horizontal if the slope equals zero, which means that $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$
2. The tangent line to the parametric curve is vertical if $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$

The second derivative is $\frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{\left(\frac{dy'}{dt}\right)}{\left(\frac{dx}{dt}\right)}$

Example 3.1

Find the slope of the tangent line(s) to the parametric curve given by

$$x = t^5 - 4t^3 \qquad y = t^2 \qquad at\ (0,4)$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{5t^4 - 12t^2} = \frac{2}{5t^3 - 12t}$$

$$0 = t^5 - 4t^3 = t^3\left(t^2 - 4\right) \qquad \Rightarrow \qquad t = 0, \pm 2$$

$$4 = t^2 \qquad \Rightarrow \qquad t = \pm 2$$

1. at $t = -2$:

$$m = \left.\frac{dy}{dx}\right|_{t=-2} = -\frac{1}{8}$$

2. at $t = 2$

$$m = \left.\frac{dy}{dx}\right|_{t=2} = \frac{1}{8}$$

Example 3.2

Find the equation of the tangent line to
$C : x = t^3 - 3t,\ y = t^2 - 5t \text{ at } t = 2$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{2t - 5}{3t^2 - 3}$$

The slope of the tangent line is $\frac{dy}{dx}\big|_{t=2} = -\frac{1}{9}$
At $t = 2 : x = 2$ and $y = -7$
The tangent line to C at $t = 2$ passes through the point $(2, -7)$ and its slope is $-\frac{1}{9}$
therefore its equation is $\frac{y+7}{x-2} = -\frac{1}{9}$

Example 3.3

Find the points on $C : x = e^t,\ y = e^{-t}$ at which the slope of the tangent line to C equals $-e^{-2}$

$m = \frac{dy}{dx} = \frac{(\frac{dy}{dt})}{(\frac{dx}{dt})} = \frac{-e^{-t}}{e^t} = -e^{-2t}$

$\Rightarrow m = e^{-2t} \Rightarrow e^{-2t} = -e^{-2} \Rightarrow t = 1.$

At $t = 1 : x = e^1 = e$ and $y = e^{-1} = \frac{1}{e}$.

Hence, the point at which the slope of the tangent line to C equals $-e^{-2}$ is $(e, \frac{1}{e})$

Arc Length of a Parametric Equations

Definition 4.1

If $C : x = x(t),\ \ y = y(t); a \leq t \leq b$ *is a differentiable parametric curve ,then its arc length equals*

$$L = \int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\ dt$$

Example 4.1

Determine the length of the parametric curve given by the following parametric equations.

$$x = 3\sin(3t) \qquad y = 3\cos(3t) \qquad 0 \le t \le 2\pi$$

$$\frac{dx}{dt} = 9\cos(3t) \qquad \frac{dy}{dt} = -9\sin(3t)$$

and the length formula gives,

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{81\sin^2(3t) + 81\cos^2(3t)}\ dt \\ &= \int_0^{2\pi} 9\ dt \\ &= 18\pi \end{aligned}$$

Example 4.2

Determine the length of the parametric curve given by the following set of parametric equations.

$$x = 8t^{\frac{3}{2}} \qquad y = 3 + (8-t)^{\frac{3}{2}} \qquad 0 \le t \le 4$$

$$\frac{dx}{dt} = 12t^{\frac{1}{2}} \qquad \frac{dy}{dt} = -\frac{3}{2}(8-t)^{\frac{1}{2}}$$

$$L = \int_0^4 \sqrt{\left[12t^{\frac{1}{2}}\right]^2 + \left[-\frac{3}{2}(8-t)^{\frac{1}{2}}\right]^2}\, dt = \int_0^4 \sqrt{144t + \frac{9}{4}(8-t)}\, dt$$

$$= \int_0^4 \sqrt{\frac{567}{4}t + 18}\, dt = \frac{4}{567}\left(\frac{2}{3}\right)\left(\frac{567}{4}t + 18\right)^{\frac{3}{2}}\Bigg|_0^4$$

$$= \frac{8}{1701}\left(585^{\frac{3}{2}} - 18^{\frac{3}{2}}\right) = 66.1865$$

Surface Area Generated By Revolving A Parametric Curve

If $C : x = x(t),\ y = y(t); a \leq t \leq b$ is a differentiable parametric curve ,then the surface area generated by revolving C around the $x-$axis is

$$SA = 2\pi \int_a^b |y(t)| \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\ dt$$

The surface area generated by revolving C around the $y-$axis is

$$SA = 2\pi \int_a^b |x(t)| \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\ dt$$

Example 5.1

Determine the surface area of the solid obtained by rotating the following parametric curve about the $x-$axis.

$$x = \cos^3\theta \qquad y = \sin^3\theta \qquad 0 \leq \theta \leq \frac{\pi}{2}$$

We'll first need the derivatives of the parametric equations.

$$\frac{dx}{d\theta} = -3\cos^2\theta \sin\theta \qquad \frac{dy}{d\theta} = 3\sin^2\theta \cos\theta$$

$$\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} = \sqrt{9\cos^4\theta\sin^2\theta + 9\sin^4\theta\cos^2\theta}\, d\theta$$

$$= 3\left|\cos\theta \sin\theta\right| \sqrt{\cos^2\theta + \sin^2\theta}$$

$$= 3\cos\theta \sin\theta$$

$$
\begin{aligned}
SA &= 2\pi \int_0^{\frac{\pi}{2}} \sin^3\theta \left(3\cos\theta \sin\theta\right)\ d\theta \\
&= 6\pi \int_0^{\frac{\pi}{2}} \sin^4\theta \cos\theta\ d\theta \qquad\qquad u = \sin\theta \\
&= 6\pi \int_0^{1} u^4\, du \\
&= \frac{6\pi}{5}
\end{aligned}
$$

Example 5.2

Determine the surface area of the object obtained by rotating the parametric curve about the y−axis.

$$x = 3\cos(\pi t) \qquad y = 5t + 2 \qquad 0 \le t \le \frac{1}{2}$$

The first thing we'll need here are the following two derivatives.

$$\frac{dx}{dt} = -3\pi\sin(\pi t) \qquad \frac{dy}{dt} = 5$$

$$\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} = \sqrt{[-3\pi\sin(\pi t)]^2 + [5]^2} = \sqrt{9\pi^2\sin^2(\pi t) + 25}$$

$$SA = \int_0^{\frac{1}{2}} 2\pi\left(3\cos(\pi t)\right)\sqrt{9\pi^2\sin^2(\pi t) + 25}\,dt$$
$$= 6\pi\int_0^{\frac{1}{2}} \cos(\pi t)\sqrt{9\pi^2\sin^2(\pi t) + 25}\,dt$$

$$u = \sin(\pi t) \quad \rightarrow \quad \sin^2(\pi t) = u^2 \qquad du = \pi\cos(\pi t)$$

$$t = 0: \quad u = \sin(0) = 0 \qquad t = \frac{1}{2}: \quad u = \sin\left(\frac{1}{2}\pi\right) = 1$$

$$SA = 6\int_0^1 \sqrt{9\pi^2 u^2 + 25}\,du$$

$$u = \frac{5}{3\pi}\tan\theta \qquad du = \frac{5}{3\pi}\sec^2\theta\, d\theta$$

$$\sqrt{9\pi^2 u^2 + 25} = \sqrt{25\tan^2\theta + 25} = 5\sqrt{\tan^2\theta + 1} = 5\sqrt{\sec^2\theta} = 5\,|\sec\theta|$$

$$u = 0 : 0 = \frac{5}{3\pi}\tan\theta \qquad \rightarrow \tan\theta = 0 \qquad \rightarrow \qquad \theta = 0$$

$$u = 1 : 1 = \frac{5}{3\pi}\tan\theta \qquad \rightarrow \tan\theta = \frac{3\pi}{5} \rightarrow \theta = \tan^{-1}\left(\frac{3\pi}{5}\right) = 1.0830$$

$$
\begin{aligned}
SA &= \int_0^{\frac{1}{2}} 2\pi\left(3\cos(\pi t)\right)\sqrt{9\pi^2\sin^2(\pi t) + 25}\,dt \\
&= 6\int_0^1 \sqrt{9\pi^2 u^2 + 25}\,du \\
&= 6\int_0^{1.0830} (5\sec\theta)\left(\frac{5}{3\pi}\sec^2\theta\right)\,d\theta \\
&= 6\int_0^{1.0830} \frac{25}{3\pi}\sec^3\theta\,d\theta \\
&= \frac{25}{\pi}\left(\sec\theta\,tan\theta + ln\left|\sec\theta + \tan\theta\right|\right)\Big|_0^{1.0830} = 43.0705
\end{aligned}
$$