INTEGRAL CALCULUS (MATH 106)

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Weekly Objectives

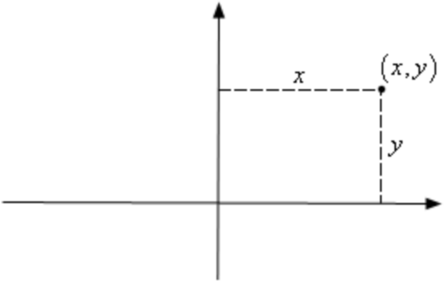
Week 13: polar coordinates

The student is expected to be able to:

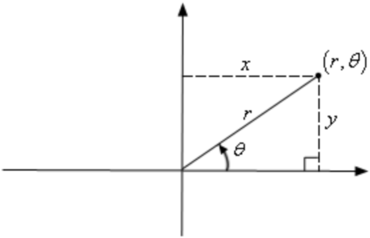
1. Know what is the polar coordinates.
2. Know the equation and graph of some of polar curves.

Cartesian coordinate system (or Rectangular, or x-y)

the Cartesian coordinate system at point is given the coordinates (x, y) and we use this to define the point by starting at the origin and then moving x units horizontally followed by y units vertically.



Cartesian coordinate is not the only way to define a point in two dimensional space. Instead of moving vertically and horizontally from the origin to get to the point we could instead go straight out of the origin until we hit the point and then determine the angle this line makes with the positive $x-$axis. We could then use the distance of the point from the origin and the amount we needed to rotate from the positive x-axis as the coordinates of the point.

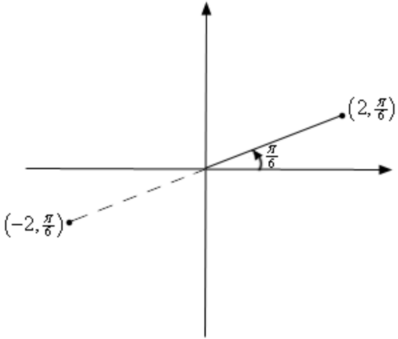


Coordinates in this form are called **polar coordinates.**

Polar Coordinates

Example 2.1

The two points $(2, \frac{\pi}{6})$ *and* $(-2, \frac{\pi}{6})$



Polar Coordinates

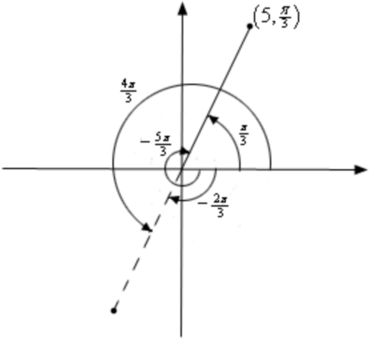
Remark

The polar coordinates of a point is not unique, if $P = (r, \theta)$ then other representations are:

1. $P = (r, \theta + 2n\pi)$, where $n \in \mathbb{Z}$
2. $P = (-r, \theta + \pi)$
3. $P = (-r, \theta + \pi + 2n\pi)$, where $n \in \mathbb{Z}$
4. $P = (-r, \theta - \pi)$
5. $P = (-r, \theta - \pi + 2n\pi)$, where $n \in \mathbb{Z}$

Example 2.2

$$\left(5, \frac{\pi}{3}\right) = \left(5, -\frac{5\pi}{3}\right) = \left(-5, \frac{4\pi}{3}\right) = \left(-5, -\frac{2\pi}{3}\right)$$



Polar Coordinates

Relationship between the polar and the Cartesian coordinates
the following equations that will convert polar coordinates into Cartesian coordinates.

$$x = r\cos\theta \qquad\qquad y = r\sin\theta$$

Converting from Cartesian is almost as easy. Let's first notice the following.

$$\begin{aligned} x^2 + y^2 &= (r\cos\theta)^2 + (r\sin\theta)^2 \\ &= r^2\cos^2\theta + r^2\sin^2\theta \\ &= r^2\left(\cos^2\theta + \sin^2\theta\right) = r^2 \end{aligned}$$

$$r = \sqrt{x^2 + y^2}, \quad \textit{and} \ \frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta$$

Example 2.3

1. *Convert* $\left(-4, \frac{2\pi}{3}\right)$ *into Cartesian coordinates.*
2. *Convert* $(-1, -1)$ *into polar coordinates.*

1. This conversion is easy enough. All we need to do is plug the points into the formulas.

$$x = -4\cos\left(\frac{2\pi}{3}\right) = -4\left(-\frac{1}{2}\right) = 2$$

$$y = -4\sin\left(\frac{2\pi}{3}\right) = -4\left(\frac{\sqrt{3}}{2}\right) = -2\sqrt{3}$$

So, in Cartesian coordinates this point is $\left(2, -2\sqrt{3}\right)$

2. Let's first get r

$r = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$

Now, let's get θ

$\theta = \tan^{-1}\left(\frac{-1}{\ }\right) = \tan^{-1}(1) = \frac{\pi}{\ }$

Example 2.4

Convert each of the following into an equation in the given coordinate system.

1. *Convert* $2x - 5x^3 = 1 + xy$ *into polar coordinates.*
2. *Convert* $r = -8\cos\theta$ *into Cartesian coordinates.*

1.
$$2\left(r\cos\theta\right) - 5(r\cos\theta)^3 = 1 + \left(r\cos\theta\right)\left(r\sin\theta\right)$$
$$2r\cos\theta - 5r^3\cos^3\theta = 1 + r^2\cos\theta\sin\theta$$

2. $r^2 = -8r\cos\theta \Rightarrow x^2 + y^2 = -8x$

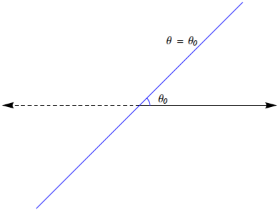
Polar Curves

First - Straight Lines:

1-Lines passing through the pole : Any straight line passing through the pole has the form $\theta = \theta_0$ where θ_0 is the angle between the straight line and the polar axis .

$\theta = \theta_0 \Rightarrow \tan\theta = \tan\theta_0 \Rightarrow \frac{y}{x} = \tan\theta_0 \Rightarrow y = x\tan\theta_0$

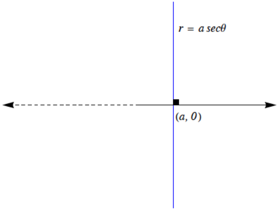
The straight line $\theta = \theta_0$ is passing through the pole with a slope equals to $\tan\theta_0$.



Polar Curves

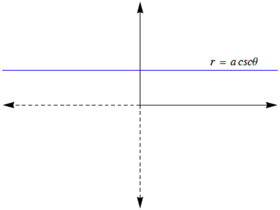
2-Lines perpendicular to the polar axis : Any straight line perpendicular to the polar axis has the form $r = a\sec\theta$, where $a \in \mathbb{R}^*$ and $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$r = a\sec\theta \Rightarrow r = \frac{a}{\cos\theta} \Rightarrow r\cos\theta = a \Rightarrow x = a$



Polar Curves

3-Lines parallel to the polar axis : Any straight line parallel to the polar axis has the form $r = a\csc\theta$, where $a \in \mathbb{R}^*$ and $\theta \in (0, \pi)$
$r = a\csc\theta \Rightarrow r = \frac{a}{\sin\theta} \Rightarrow r\sin\theta = a \Rightarrow y = a$



Polar Curves

Second- Circles

1-Circles of the form $r = a$, where $a \in \mathbb{R}^*$

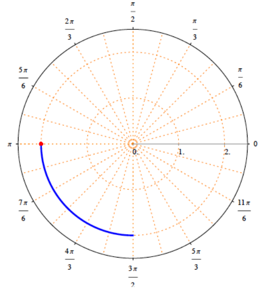
$r = a \Rightarrow r^2 = a^2 \Rightarrow x^2 + y^2 = a^2$

Therefore, $r = a$ represents a circle with center $= (0, 0)$ and radius equals $|a|$

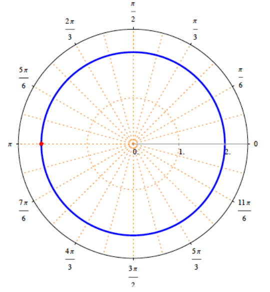
Polar Curves

Example 3.1

$r = -2$ represents a circle with center $= (0, 0)$ and radius to 2.



$r = -2 \ , \ 0 \leq \theta \leq \frac{\pi}{2}$

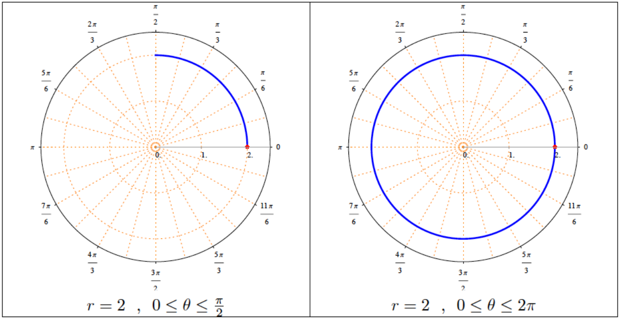


$r = -2 \ , \ 0 \leq \theta \leq 2\pi$

Polar Curves

Example 3.2

$r = 2$ represents a circle with center $= (0, 0)$ and radius to 2.



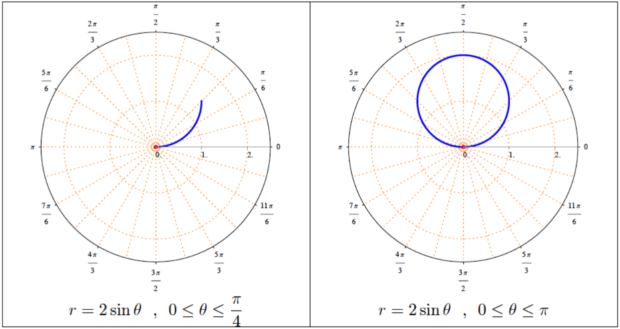
Polar Curves

2-Circles of the form $r = a\sin\theta$, where $a \in \mathbb{R}^*$ and $0 \leq \theta \leq \pi$
$r = a\sin\theta \Rightarrow r^2 = a\ r\sin\theta \Rightarrow x^2 + y^2 = ay \Rightarrow x^2 + y^2 - ay = 0 \Rightarrow x^2 + (y^2 - ay + \frac{a^2}{4}) = \frac{a^2}{4} \Rightarrow x^2 + (y - \frac{a}{2})^2 = \frac{a^2}{4}$
Therefore, $r = a\sin\theta$ represents a circle with center $= (0, \frac{a}{2})$ and radius equals to $\frac{|a|}{2}$

Polar Curves

Example 3.3

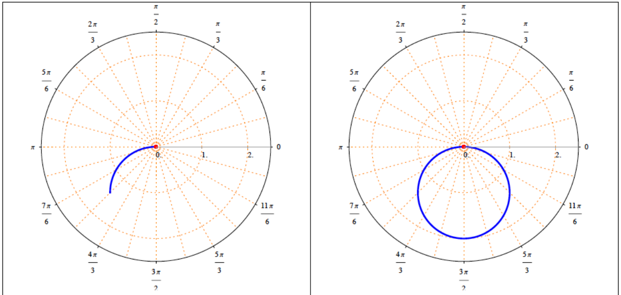
$r = 2\sin\theta$ *represents a circle with center* $= (0, 1)$ *and radius equals to 1*



Polar Curves

Example 3.4

$r = -2\sin\theta$ represents a circle with center $= (0, -1)$ and radius equals to 1.



$r = -2\sin\theta \;,\; 0 \le \theta \le \frac{\pi}{4}$ $\qquad$ $r = -2\sin\theta \;,\; 0 \le \theta \le \pi$

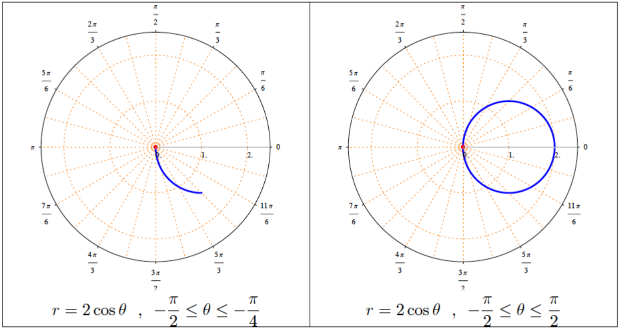
Polar Curves

3-Circles of the form $r = a\cos\theta$, where $a \in \mathbb{R}^*$ and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$
$r = a\cos\theta \Rightarrow r^2 = a\ r\cos\theta \Rightarrow x^2 + y^2 = ax \Rightarrow x^2 - ax + y^2 = 0 \Rightarrow (x^2 - ax + \frac{a^2}{4}) + y^2 = \frac{a^2}{4} \Rightarrow (x - \frac{a}{2})^2 + y^2 = \frac{a^2}{4}$
Therefore, $r = a\cos\theta$ represents a circle with center $= (\frac{a}{2}, 0)$ and radius equals to $\frac{|a|}{2}$

Polar Curves

Example 3.5

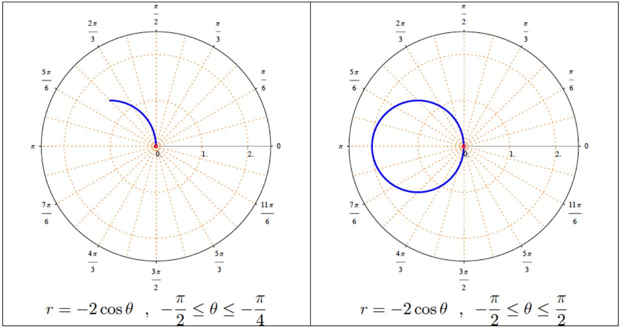
$r = 2\cos\theta$ represents a circle with center $= (1, 0)$ and radius equals to 1.



Polar Curves

Example 3.6

$r = -2\cos\theta$ represents a circle with center $= (-1, 0)$ and radius equals to 1.



Polar Curves

Third - Limacon curves:

The general form of a Limacon curve is

$r(\theta) = a + b\sin\theta$ or $r(\theta) = a + b\cos\theta$, where $a, b \in \mathbb{R}^*$ and $0 \leq \theta \leq 2\pi$

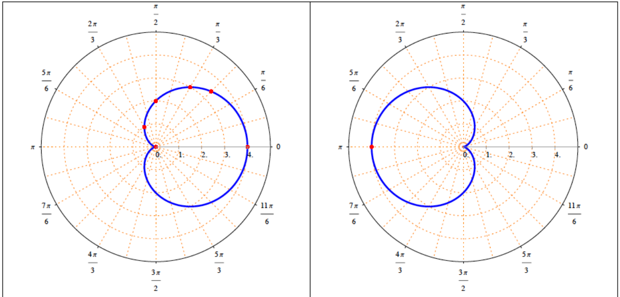
1-Cardioid (Heart-shaped): It has the form $r(\theta) = a + a\sin\theta$ or $r(\theta) = a + a\cos\theta$, where $a \in \mathbb{R}^*$ and $0 \leq \theta \leq 2\pi$

Polar Curves

Example 3.7

$r(\theta) = 2 + 2\cos\theta$

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	4	$2+\sqrt{2}$	3	2	1	0

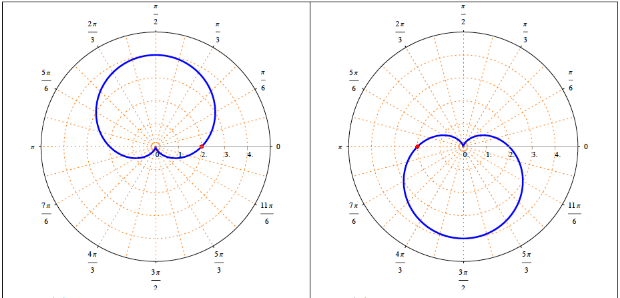


$r(\theta) = 2 + 2\cos\theta \quad , \quad 0 \le \theta \le 2\pi$ $\qquad$ $r(\theta) = -2 - 2\cos\theta \quad , \quad 0 \le \theta \le 2\pi$

Polar Curves

Example 3.8

$r(\theta) = 2 + 2\sin\theta$ *and* $r(\theta) = -2 - 2\sin\theta$



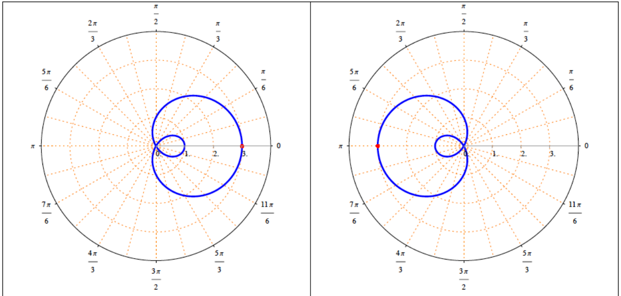
$r(\theta) = 2 + 2\sin\theta \;,\; 0 \le \theta \le 2\pi$ $\qquad$ $r(\theta) = -2 - 2\sin\theta \;,\; 0 \le \theta \le 2\pi$

Polar Curves

2-Limacon with inner loop: It has the form $r(\theta) = a + b\sin\theta$ or $r(\theta) = a + b\cos\theta$, where $a, b \in \mathbb{R}^*$, $|a| < |b|$ and $0 \le \theta \le 2\pi$

Example 3.9

$r(\theta) = 1 + 2\cos\theta$ *and* $r(\theta) = -1 - 2\cos\theta$

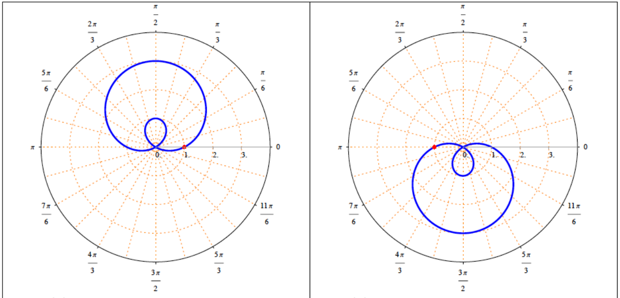


$r(\theta) = 1 + 2\cos\theta \quad , \quad 0 \le \theta \le 2\pi$ $\qquad$ $r(\theta) = -1 - 2\cos\theta \quad , \quad 0 \le \theta \le 2\pi$

Polar Curves

Example 3.10

$r(\theta) = 1 + 2\sin\theta$ *and* $r(\theta) = -1 - 2\sin\theta$



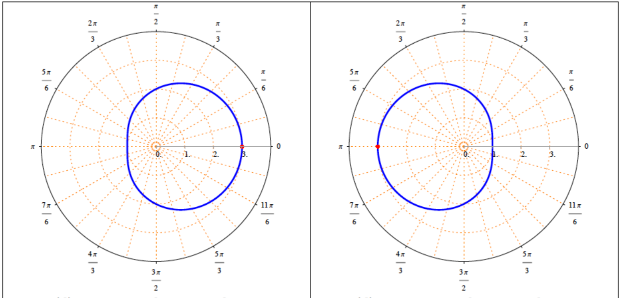
$r(\theta) = 1 + 2\sin\theta \quad , \quad 0 \le \theta \le 2\pi$ $\qquad$ $r(\theta) = -1 - 2\sin\theta \quad , \quad 0 \le \theta \le 2\pi$

Polar Curves

3-Dimpled Limacon : It has the form $r(\theta) = a + b\sin\theta$ or $r(\theta)a + b\cos\theta$, where $a, b \in \mathbb{R}^*$, $|a| > |b|$ and $0 \leq \theta \leq 2\pi$

Example 3.11

$r(\theta) = 2 + \cos\theta$ *and* $r(\theta) = -2 - \cos\theta$

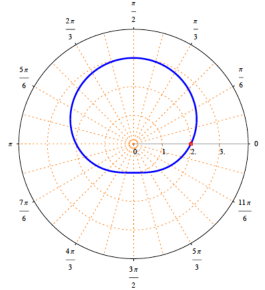


$r(\theta) = 2 + \cos\theta \;\;,\;\; 0 \leq \theta \leq 2\pi$ $\qquad$ $r(\theta) = -2 - \cos\theta \;\;,\;\; 0 \leq \theta \leq 2\pi$

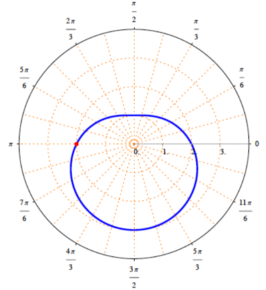
Polar Curves

Example 3.12

$r(\theta) = 2 + \sin\theta$ *and* $r(\theta) = -2 - \sin\theta$



$r(\theta) = 2 + \sin\theta \quad , \quad 0 \le \theta \le 2\pi$



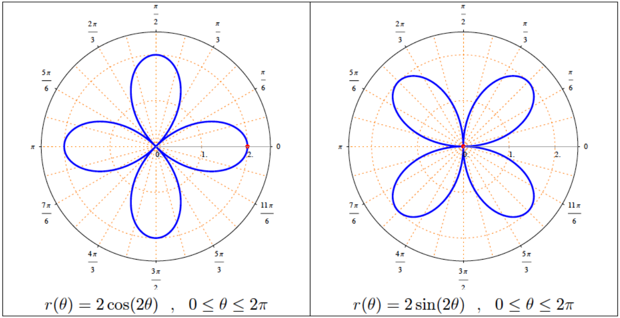
$r(\theta) = -2 - \sin\theta \quad , \quad 0 \le \theta \le 2\pi$

Polar Curves

Fourth - Rose curves:
It has the form $r(\theta) = a\cos(n\theta)$ or $r(\theta) = a\sin(n\theta)$, where $a \in \mathbb{R}^*, n \in \mathbb{N}$ and $n \geq 2$
1-n is even: In this case the number of loops (or leaves) is 2n.

Example 3.13

$r(\theta) = 2\cos(2\theta)$ *or* $r(\theta) = 2\sin(2\theta), 0 \leq \theta \leq 2\pi$



Polar Curves

n is odd: In this case the number of loops (or leaves) is n .

Example 3.14

$r(\theta) = 2\cos(3\theta)$ *or* $r(\theta) = 2\sin(3\theta), 0 \leq \theta \leq \pi$