INTEGRAL CALCULUS (MATH 106)

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1. Slope Of The Tangents Line With Polar Coordinates
2. Area Inside-Between Polar Curves
3. Arc Length Of A Polar Curve
4. Surface Area Generated By Revolving A Polar Curve

Weekly Objectives

Week 14: polar coordinates

The student is expected to be able to:

1. Know how to calculate the slope of the tangent line with polar coordinates.
2. Know how to calculate the area inside polar curves and between polar curves.
3. Know how to calculate the arc length of a polar curve.
4. Know how to calculate Surface Area Generated By Revolving A Polar Curve.

Slope Of The Tangents Line With Polar Coordinates

If $r = r(\theta)$ is a smooth polar curve, then the slope of the tangent line to $r = r(\theta)$ is $m = \frac{dy}{dx}$ where $(x = r\cos\theta, \quad y = r\sin\theta)$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\dfrac{dr}{d\theta}\sin\theta + r\cos\theta}{\dfrac{dr}{d\theta}\cos\theta - r\sin\theta}$$

Example

Example 2.1

Determine the equation of the tangent line to $r = 3 + 8\sin\theta$ *at* $\theta = \dfrac{\pi}{6}$

We'll first need the following derivative. $\frac{dr}{d\theta} = 8\cos\theta$
The formula for the derivative $\frac{dy}{dx}$ becomes,

$$\frac{dy}{dx} = \frac{8\cos\theta\sin\theta + (3 + 8\sin\theta)\cos\theta}{8\cos^2\theta - (3 + 8\sin\theta)\sin\theta} = \frac{16\cos\theta\sin\theta + 3\cos\theta}{8\cos^2\theta - 3\sin\theta - 8\sin^2\theta}$$

The slope of the tangent line is,

$$m = \left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{6}} = \frac{4\sqrt{3} + \frac{3\sqrt{3}}{2}}{4 - \frac{3}{2}} = \frac{11\sqrt{3}}{5}$$

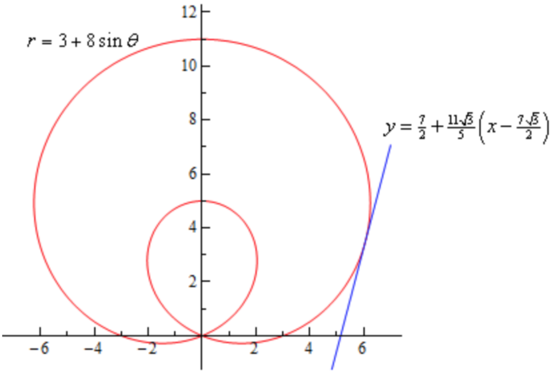
Now, at $\theta = \frac{\pi}{6}$ we have $r = 7$ We'll need to get the corresponding $x - y$ coordinates so we can get the tangent line.

$$x = 7\cos\left(\frac{\pi}{6}\right) = \frac{7\sqrt{3}}{2} \qquad y = 7\sin\left(\frac{\pi}{6}\right) = \frac{7}{2}$$

The tangent line is then,

$$y = \frac{7}{2} + \frac{11\sqrt{3}}{5}\left(x - \frac{7\sqrt{3}}{2}\right)$$

For the sake of completeness here is a graph of the curve and the tangent line.



Example 2.2

Find the points on the polar curve $r(\theta) = 1 + \cos\theta,\ 0 \le \theta \le 2\pi$ at which the tangent line to r is horizontal.

The tangent line to $r = r(\theta)$ is horizontal if $\frac{dy}{d\theta} = 0$ and $\frac{dx}{d\theta} \neq 0$

$x = r(\theta)\cos\theta \Rightarrow x = \cos\theta(1 + \cos\theta) = \cos\theta + \cos^2\theta$

$y = r(\theta)\sin\theta \Rightarrow y = \sin\theta(1 + \cos\theta) = \sin\theta + \sin\theta\cos\theta = \sin\theta + \frac{1}{2}\sin 2\theta$

$\frac{dx}{d\theta} = -\sin\theta - 2\cos\theta\sin\theta = -\sin\theta - \sin 2\theta$

$\frac{dy}{d\theta} = \cos\theta + \cos 2\theta$

$\frac{dy}{d\theta} = 0 \Rightarrow \cos\theta + \cos 2\theta = 0 \Rightarrow 2\cos^2\theta - 1 + \cos\theta = 0 \Rightarrow (2\cos\theta - 1)(\cos\theta + 1) = 0 \Rightarrow \cos\theta = -1$ or $\cos\theta = \frac{1}{2}$

$\Rightarrow \theta = \pi$ or $\theta = \frac{\pi}{3}, \theta = \frac{5\pi}{3}$

For $\theta = \pi$, $\frac{dx}{d\theta} = 0$.
For $\theta = \frac{\pi}{3}, \theta = \frac{5\pi}{3} \in [0, 2\pi]$ and $\frac{dx}{d\theta} \neq 0$.
At $\theta = \frac{\pi}{3} : r(\pi 3) = 1 + \frac{1}{2} = \frac{3}{2}$
At $\theta = \frac{5\pi}{3} : r(\frac{5\pi}{3}) = 1 + \frac{1}{2} = \frac{3}{2}$
The points on $r(\theta) = 1 + \cos\theta, 0 \leq \theta \leq 2\pi$ at which the tangent line to r is horizontal are $(\frac{3}{2}, \frac{\pi}{3}), (\frac{3}{2}, \frac{5\pi}{3})$

Area Inside-Between Polar Curves

The area of the region bounded by the graphs of the polar curves $r = r(\theta), \theta = \theta_1$ and $\theta = \theta_2$ is

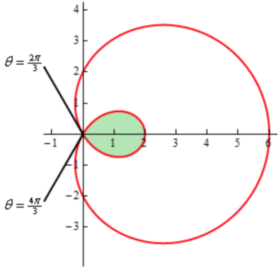
$$A = \frac{1}{2} \int_{\theta_1}^{\theta_2} [r(\theta)]^2 \, d\theta$$

Example 3.1

Determine the area of the inner loop of $r = 2 + 4\cos\theta$

$$0 = 2 + 4\cos\theta$$

$$\cos\theta = -\frac{1}{2} \qquad \Rightarrow \qquad \theta = \frac{2\pi}{3}, \frac{4\pi}{3}$$



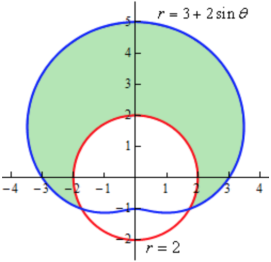
$$
\begin{aligned}
A &= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{1}{2}(2+4\cos\theta)^2\,d\theta \\
&= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{1}{2}\left(4+16\cos\theta+16\cos^2\theta\right)\,d\theta \\
&= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} 2+8\cos\theta+4\left(1+\cos(2\theta)\right)\,d\theta \\
&= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} 6+8\cos\theta+4\cos(2\theta)\,d\theta \\
&= \left(6\theta+8\sin\theta+2\sin(2\theta)\right)\Big|_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \\
&= 4\pi-6\sqrt{3}=2.174
\end{aligned}
$$

Example 3.2

Determine the area that lies inside $r = 3 + 2\sin\theta$ and outside $r = 2$

$$3 + 2\sin\theta = 2$$

$$\sin\theta = -\frac{1}{2} \quad \Rightarrow \quad \theta = \frac{7\pi}{6}, \frac{11\pi}{6}$$



$$A = \int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} \frac{1}{2}\left((3+2\sin\theta)^2 - (2)^2\right)\, d\theta$$

$$= \int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} \frac{1}{2}\left(5+12\sin\theta + 4\sin^2\theta\right)\, d\theta$$

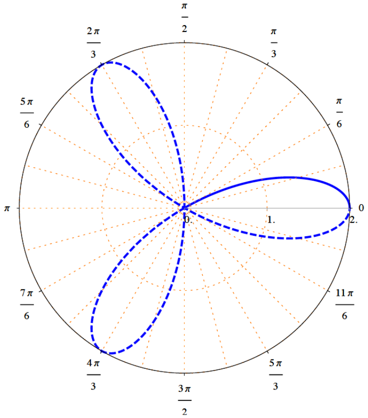
$$= \int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} \frac{1}{2}\left(7+12\sin\theta - 2\cos(2\theta)\right)\, d\theta$$

$$= \frac{1}{2}\left(7\theta - 12\cos\theta - \sin(2\theta)\right)\Big|_{-\frac{\pi}{6}}^{\frac{7\pi}{6}}$$

$$= \frac{11\sqrt{3}}{2} + \frac{14\pi}{3} = 24.187$$

Example 3.3

Find the area inside one leaf of the rose curve $r = 2\cos 3\theta$



The rose curve $r = 2\cos 3\theta, 0 \le \theta \le \pi$ starts at $(r, \theta) = (2, 0)$ and reaches the pole when $r = 0$
$r = 0 \Rightarrow 2\cos 3\theta = 0 \Rightarrow 3\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{6}$ Since the desired area is symmetric with respect to the polar axis , then

$$\begin{aligned}
A &= 2\left(\frac{1}{2}\int_0^{\frac{\pi}{6}} (2\cos 3\theta)^2 d\theta\right) \\
&= 4\int_0^{\frac{\pi}{6}} \cos^2 3\theta \, d\theta \\
&= 4\int_0^{\frac{\pi}{6}} \frac{1}{2}(1+\cos 6\theta)\, d\theta \\
&= 2\int_0^{\frac{\pi}{6}} (1+\cos 6\theta)\, d\theta \\
&= 2\left[\theta + \frac{\sin 6\theta}{6}\right]_0^{\frac{\pi}{6}} = \frac{\pi}{3}
\end{aligned}$$

Arc Length Of A Polar Curve

The arc length of the polar curve $r = r(\theta)$ from θ_1 to θ_2 is

$$L = \int_{\theta_1}^{\theta_2} \sqrt{(r(\theta))^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Example 4.1

Determine the length of the following polar curve.
$r = -4\sin\theta,\ 0 \le \theta \le \pi$

$\frac{dr}{d\theta} = -4\cos\theta$

$$L == \int_0^\pi \sqrt{[-4\sin\theta]^2 + [-4\cos\theta]^2}\, d\theta$$

$$= \int_0^\pi \sqrt{16\sin^2\theta + 16\cos^2\theta}\, d\theta = 4\sqrt{\sin^2\theta + \cos^2\theta}\, d\theta = \int_0^\pi 4 d\theta$$

$$L = \int_0^\pi 4\, d\theta = [4\theta]_0^\pi = 4\pi$$

Example 4.2

Find the arc length of the following polar curve: $r = e^{-\theta}$

$$\frac{dr}{d\theta} = -e^{-\theta}$$

$$L == \int_0^{\pi} \sqrt{(e^{-\theta})^2 + (-e^{-\theta})^2}\, d\theta$$

$$= \int_0^{\pi} \sqrt{e^{-2\theta} + e^{-2\theta}}\, d\theta = \int_0^{\pi} \sqrt{2e^{-2\theta}}\, d\theta = \sqrt{2}\int_0^{\pi} e^{-\theta}\, d\theta$$

$$L = \sqrt{2}\left[-e^{-\theta}\right]_0^{\pi} = \sqrt{2}[-e^{-\pi} + e^0] = \sqrt{2}(1 - e^{-\pi})$$

Surface Area Generated By Revolving A Polar Curve

The surface area generated by revolving the polar curve $r = r(\theta),\ \theta_1 \leq \theta \leq \theta_2$ around the polar axis is

$$SA = 2\pi \int_{\theta_1}^{\theta_2} |r(\theta)\sin\theta| \sqrt{(r(\theta))^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

The surface area generated by revolving the polar curve $r = r(\theta),\ \theta_1 \leq \theta \leq \theta_2$ around the line $\theta = \frac{\pi}{2}$ is

$$SA = 2\pi \int_{\theta_1}^{\theta_2} |r(\theta)\cos\theta| \sqrt{(r(\theta))^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Example 5.1

Find the surface area generated by revolving the following polar curve: $r = 2 + 2\cos\theta, 0 \le \theta \le \frac{\pi}{2}$ *around the polar axis.*

$\frac{dr}{d\theta} = -2\sin\theta$

$$SA = 2\pi \int_0^{\frac{\pi}{2}} |(2+2\cos\theta)\sin\theta| \sqrt{(2+2\cos\theta)^2 + (-2\sin\theta)^2}\, d\theta$$

$$= 2\pi \int_0^{\frac{\pi}{2}} (2+2\cos\theta)\sin\theta \sqrt{4\,(2+2\cos\theta)}\, d\theta$$

$$= 4\pi \int_0^{\frac{\pi}{2}} (2+2\cos\theta)^{\frac{3}{2}} \sin\theta\, d\theta$$

$$SA = -2\pi \int_{0}^{\frac{\pi}{2}} (2+2\cos\theta)^{\frac{3}{2}} \, (-2\sin\theta) \; d\theta$$

$$= -2\pi \left[\frac{2}{5} (2+2\cos\theta)^{\frac{5}{2}} \right]_{0}^{\frac{\pi}{2}}$$

$$= -2\pi \frac{2}{5} \left[4\sqrt{2} - 32 \right] = \frac{16\pi}{5} \left(8 - \sqrt{2} \right)$$

Example 5.2

Find the surface area generated by revolving the following polar curve: $r = 2\sin\theta, 0 \leq \theta \leq \frac{\pi}{2}$ *around the line* $\theta = \frac{\pi}{2}$

$\frac{dr}{d\theta} = 2\cos\theta$

$$SA = 2\pi \int_0^{\frac{\pi}{2}} |2\sin\theta\cos\theta| \sqrt{(2\sin\theta)^2 + (2\cos\theta)^2}\, d\theta$$

$$= 2\pi \int_0^{\frac{\pi}{2}} \sin 2\theta \sqrt{4}\, d\theta$$

$$SA = 4\pi \left[-\frac{\cos 2\theta}{2}\right]_0^{\frac{\pi}{2}} = 4\pi$$