INTEGRAL CALCULUS (MATH 106)

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Weekly Objectives

Week 2: Summation Notation and definite Integrals.

The student is expected to be able to:

1. Become familiar with the summation notation.
2. Know the definition of definite integral with Riemann Sum.
3. Properties of the definite integral.

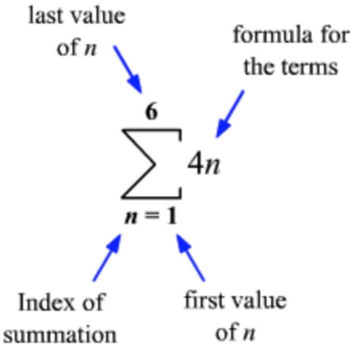
Sums and sigma notation

- A series can be represented in a compact form, called summation or sigma notation.
- The Greek capital letter, $\sum$, is used to represent the sum.
- The series $4 + 8 + 12 + 16 + 20 + 24$ can be expressed as

$$\sum_{n=1}^{6} 4n$$

- The expression is read as the sum of 4n as n goes from 1 to 6 .
- The variable n is called the index of summation.

Sums and sigma notation



So if $a_1, a_2, \ldots, a_n \in \mathbb{R}$, then $\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$

Sums and sigma notation

Theorem 2.1

If $c, a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n \in \mathbb{R}$, *Then,*

1. $\sum_{i=1}^{n} C = Cn$
2. $\sum_{i=1}^{n} Ca_i = C \sum_{i=1}^{n} a_i$
3. $\sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$
4. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
5. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
6. $\sum_{i=1}^{n} i^3 = (\frac{n(n+1)}{2})^2$

Sums and sigma notation (Examples)

Example 2.1

Use the properties in the theorem 2.1 to find the value of:

$$\sum_{i=1}^{4}(k^3 - k + 2)$$

Solution 1

$$\sum_{i=1}^{4}(k^3 - k + 2) = \sum_{i=1}^{4} k^3 - \sum_{i=1}^{4} k + \sum_{i=1}^{4} 2 = \sum_{i=1}^{4} k^3 - \sum_{i=1}^{4} k + 2\sum_{i=1}^{4}$$
$$= \left(\frac{4(4+1)}{2}\right)^2 - \frac{4(4+1)}{2} + (2 \times 4) = 98$$

Sums and sigma notation (Exercises)

Exercise 1

Using the formulas and properties from above determine the value of the following summations.

1. $\sum_{i=1}^{100} (3-2i)^2$ 1293700
2. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n^3}(i-1)^2$ $\frac{1}{3}$
3. $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{5k}{n^2}$ $\frac{5}{2}$

Riemann Sum: Regular Partition

In this section we assume that the function $f(x) \geq 0$ on the interval $[a, b]$.

Definition 3.1

The set $\{a = x_0, x_1, \ldots, x_n = b\}$ *is called a* **regular partition** *of the interval* $[a, b]$
if $x_i = x_0 + i\triangle x$ *for every* $i = 1, 2, \ldots, n$, *and* $\triangle x = \frac{b-a}{n}$.
This regular partition divides the interval $[a, b]$ *into* n *subintervals of the form* $[x_{i-1}, x_i]$ *where* $i = 1, 2, \ldots, n$.

Riemann Sum: Area under the graph of a function

- Area under the graph of a function :

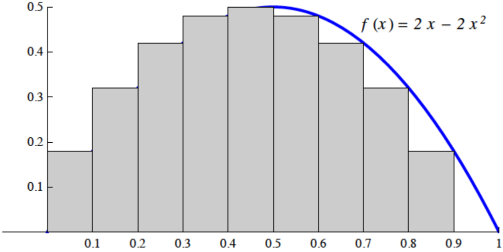
If $f(x) \geq 0$ on the interval $[a, b]$ and $\{x_0 = a, x_1, \ldots, x_n = b\}$ is a regular partition of $[a, b]$, then the area under the graph of $f(x)$ can be approximated by n rectangles using the formula:

$$A_n = \sum_{k=1}^{n} f(x_i) \triangle x$$

Riemann Sum: Exemple

Example 3.1

Approximate the area under the graph of $f(x) = 2x - 2x^2$ *on the interval* $[0, 1]$ *using 10 rectangles .*



Riemann Sum: Exemple

solution

1. $\triangle x = \frac{1-0}{10} = 0.1$
2. $x_0 = 0, x_1 = 0.1, x_2 = 0.2, \ldots, x_9 = 0.9, x_{10} = 1$
3. $A_{10} = \sum_{i=1}^{10} f(x_i)\triangle x = \sum_{i=1}^{10} (2x_i - 2x_i^2)0.1$
4. $A_{10} =$ $0.1[0.18+0.32+0.42+0.48+0.5+0.48+0.42+0.32+0.18+0]$
5. $A_{10} = 0.1(3.3) = 0.33$

Riemann Sum: Definition

Definition 3.2

Let $\{x_0 = a, x_1, \ldots, x_n = b\}$ *be a* **regular partition** *of the interval* $[a, b]$ *with* $\triangle x = \frac{b-a}{n}$. *Pick points* $c_1, c_2, \ldots, c_n$ *where* c_i *is any point in the subintrval* $[x_{i-1}, x_i], i = 1, 2, \ldots, n$.
The Riemann sum is:

$$R_n = \sum_{i=1}^{n} f(c_i)\triangle x$$

The area under the curve of $f(x)$ *is the limit of the Riemann sum.*

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\triangle x$$

Riemann Sum: Exemple

Example 3.2

Find the area under the curve of the function $f(x) = 3x + 1$ *on the interval* $[1, 3]$ *using Riemann sum and* c_i *is the middle point of the subinterval.*

Riemann Sum: Exemple

Solution

1. $\triangle x = \frac{b-a}{n} = \frac{2}{n}$
2. $x_0 = 1, x_i = x_0 + i\triangle x = 1 + \frac{2i}{n}$ for every $i = 1, 2, \ldots, n$
3. For every $i = 1, 2, \ldots, n, c_i \in [x_{i-1}, x_i], c_i = \frac{x_i + x_{i-1}}{2} = 1 + \frac{2i-1}{n}.$
4. $R_n = \sum_{i=1}^{n} f(c_i)\triangle x = \sum_{i=1}^{n} [3(1 + \frac{2i-1}{n}) + 1]\frac{2}{n} = 8 + 6\frac{n(n+1)}{n^2} - \frac{6}{n}.$
5. The desired area A is: $A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} 8 + 6\frac{n(n+1)}{n^2} - \frac{6}{n} = 14$

Riemann Sum: Exercise

Exercise 2

Do the last example where c_i is the end point of the subinterval.

The definite Integral: Definition

Definition 4.1

For any continuous function f defined on the interval $[a, b]$ the definite integral of f from a to b is:

$$\int_a^b f(x)dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\triangle x,$$

whenever the limit exists.
(where c_i is any point in the subintrval $[x_{i-1}, x_i], i = 1, 2, \ldots, n$).

The definite Integral: Remarks

Remark 4.1

1. *Rieman Sum is the same for any choice of the points* $c_1, c_2, \ldots, c_n$
2. *When the limit exists we say that the function* f *is integrable.*

Remark 4.2

If the function f *is continuous on* $[a, b]$ *and* $f(x) \geq 0$ *for every* $x \in [a, b]$ *, then*

1. $\int\limits_a^b f(x)dx \geq 0$
2. $\int\limits_a^b f(x)dx =$ *The area under the curve of f*

The definite Integral: Example

Example 4.1

$\int_1^3 (3x+1)dx = Area\ under\ the\ curve\ of\ f$

and from 3.2 we have $A = \lim_{n\to\infty} R_n = 14$, *so*

$$\int_1^3 (3x+1)dx = 14$$

The definite Integral: Exercises

Exercise 3

Estimate the area of the region between the function and the x-axis on the given interval using $n = 6$ and using, the midpoints of the subintervals for the height of the rectangles.

1. $f(x) = x^3 - 2x^2 + 4$ *on* $[1, 4]$ — $A = 33.40625$
2. $g(x) = 4 - \sqrt{x^2 + 2}$ *on* $[-1, 3]$ — $A = 8.031494$
3. $h(x) = -x\cos\left(\frac{x}{3}\right)$ *on* $[0, 3]$ — $A = -3.449532$

Observation

In the last exercise, do not get excited about the negative area here. As we discussed in this section this just means that the graph, in this case, is below the x-axis.

The definite Integral: Theorem

Theorem 4.1

If the function f is continuous on the interval $[a, b]$ then f is integrable on $[a, b]$.

The definite Integral: Properties of the definite integral

1. $\int_b^a kf(x)dx = k\int_b^a f(x)dx$ for every $k \in \mathbb{R}$.
2. $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
3. For every $c \in [a, b]$, $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$
4. If $f(x) \leq g(x)$ for every $x \in [a, b]$,

   then $\int_a^b f(x)dx \leq \int_a^b g(x)dx$
5. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

The definite Integral (Examples)

1. $$\int_7^2 3(x^2-3)dx = 3\int_7^2 (x^2-3)dx = -3\int_2^7 (x^2-3)dx = -290$$
2. $$\int_7^2 3(x^2-3)dx = -3\int_2^7 (x^2-3)dx =$$
   $$-3\int_2^5 (x^2-3)dx - 3\int_5^7 (x^2-3)dx = -290$$
3. $x^2 \geq \frac{x^2}{x^2+4}$ then, $\int_{-1}^1 x^2 dx \geq \int_{-1}^1 \frac{x^2}{x^2+4}dx$