INTEGRAL CALCULUS (MATH 106)

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1. Fundamental Theorem of Calculus and Numerical Integration
2. Average value of a function
3. Numerical Integration
    - Trapezoidal Rule
    - Simpson's Rule

Weekly Objectives

Week 3: Fundamental Theorem of Calculus and Numerical Integration.

The student is expected to be able to:

1. Compute definite integrals using the Fundamental Theorem of Calculus.
2. Approximates definite integrals with different methods such as Trapozoidal, Simson.

Fundamental Theorem of Calculus (Part I)

If f is a continuous function on the interval $[a, b]$, and $G(x)$ is the antiderivative of $f(x)$ on $[a, b]$ then:

$$\int_a^b f(x)dx = [G(x)]_a^b = G(b) - G(a)$$

Remark

$$\int_a^b \frac{d}{dx} G(x)dx = G(b) - G(a)$$

Example 2.1

1. $$\int_0^2 (x^2 - 2x)dx = \left[\frac{x^3}{3} - x^2\right]_0^2 = \left(\frac{8}{3} - 4\right) - \left(\frac{0}{3} - 0\right) = -\frac{4}{3}$$
2. *Find the area under the graph of* $f(x) = \sin x$, *on* $[0, \pi]$.
   The area:
   $$A = \int_0^\pi \sin x \; dx = [-\cos x]_0^\pi = (-\cos \pi) - (-\cos 0) = 2$$

Fundamental Theorem of Calculus (Part II)

If f is a continuous function on the interval $[a, b]$ and $G(x) = \int_a^x f(t)dt$ for every $x \in [a, b]$ then $G'(x) = f(x)$ for every $x \in [a, b]$.

Example 2.2

1. $\frac{d}{dx} \int_0^x \sqrt{t^2 + 1}\, dt = \sqrt{x^2 + 1}$
2. $\frac{d}{dx} \int_1^x \frac{1}{t^2 + 1}\, dt = \frac{1}{x^2 + 1}$

Fundamental Theorem of Calculus

Theorem 2.1

If f is a continuous function , g and h are deifferentiable functions then

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = f(h(x))h'(x) - f(g(x))g'(x)$$

Fundamental Theorem of Calculus: Example

Example 2.3

Find $G'(x)$, *if* $G(x) = \int_{1-x}^{x^2} \frac{1}{4+3t^2} dt$

$$G'(x) = \frac{d}{dx} \int_{1-x}^{x^2} \frac{1}{4+3t^2}\, dt =$$

$$= \frac{1}{4+3(x^2)^2}(2x) - \frac{1}{4+3(1-x)^2}(-1)$$

$$G'(x) = \frac{2x}{4+3(x^2)^2} + \frac{1}{4+3(1-x)^2}$$

Fundamental Theorem of Calculus

Remark 2.1

1. *If* $g(x) = a$ *and* $h(x) = b$ *then*
$$\frac{d}{dx}\int_a^b f(t)\,dt = f(b)(0) - f(a)(0) = 0$$
2. *If* $g(x) = a$ *and* $h(x) = x$ *then*
$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)(1) - f(a)(0) = f(x)$$

Fundamental Theorem of Calculus: Example

Example 2.4

Find $F'(2)$, *if* $F(x) = \int_{1}^{x^2} \frac{1}{t} dt.$

Solution

$$F'(2) = \frac{d}{dx} \int_{1}^{x^2} \frac{1}{t} dt \,\Big|_{x=2} = \left(\frac{1}{x^2}(2x) - 0 \right)_{x=2} = \left(\frac{2x}{x^2} \right)_{x=2} = \frac{2}{2} = 1$$

Fundamental Theorem of Calculus: Example

Example 2.5

Find the derivative of $F(x) = \int_2^{x^2} \ln(t)\, dt.$

Solution

From the theorem 2.1 we have:

$$\frac{d}{dx} \int_{g(x)}^{h(x)} f(t)\, dt = f(h(x))h'(x) - f(g(x))g'(x)$$

$$F'(x) = \frac{d}{dx} \int_2^{x^2} \ln(t)\, dt = \ln(h(x))h'(x) - 0$$

where $h(x) = x^2$, so, we find $F'(x) = \ln(x^2)2x = 2x\ln(x^2)$

Fundamental Theorem of Calculus: Example

Example 2.6

Find the derivative of $F(x) = \int_{\cos(x)}^{5} t^3 \, dt$.

Solution

From the theorem 2.1 we have:

$$\frac{d}{dx} \int_{g(x)}^{h(x)} f(t) \, dt = f(h(x))h'(x) - f(g(x))g'(x)$$

$$F'(x) = \frac{d}{dx} \int_{\cos(x)}^{5} t^3 \, dt = 0 - f(g(x))g'(x)$$

Where $g(x) = \cos x$, so we find:

Average value of a function: Definition

Definition 3.1

Let f be a continuous function on [a, b] then the average value of f on [a, b] is

$$f_{av} = \frac{\int_a^b f(x)dx}{b - a}$$

Average value of a function: Example

Example 3.1

Find f_{av} of the following function: $f(x) = x^2 - 2x$ *on the interval* $[1, 4]$

$$\int_1^4 (x^2 - 2x)dx = \left[\frac{x^3}{3} - x^2\right]_1^4 = 6$$

Hence $f_{av} = \frac{\int_1^4 (x^2 - 2x)dx}{4-1} = \frac{6}{3} = 2$

Average value of a function: Exercises

Exercise 1

1. *Find* f_{av} *of the function* $f(x) = (2x+1)^2$ *on the interval* $[0, 1]$
2. *Find* f_{av} *of the function* $f(x) = \sin^2 x \cos x$ *on the interval* $[0, \frac{\pi}{2}]$

Integral Mean Value Theorem

Theorem 3.1

If f is a continuous function on the interval $[a, b]$ then there exists a number $c \in (a, b)$ for which $f(c) = \frac{\int_a^b f(x)dx}{b-a}$

Example 3.2

Find the value that satisfies the integral Mean value theorem for the function $f(x) = 4x^3 - 1$ *on the interval* $[1, 2]$

$f(c) = \frac{\int_1^2 (4x^3-1)}{2-1} \Rightarrow 4c^3-1 = [x^4-x]_1^2 \Rightarrow 4c^3-1 = 14 \Rightarrow c = \sqrt[3]{\frac{15}{4}}$

Note that $\sqrt[3]{\frac{15}{4}} \in [1, 2]$

Numerical Integration:Trapezoidal Rule

The Trapezoidal Rule: It is used to approximate $\int_a^b f(x)dx$ with a regular partition of the interval $[a, b]$, where $\triangle x = \frac{b-a}{n}$, by using the formula

$$\int_a^b f(x)dx \approx \frac{b-a}{2n}[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

Numerical Integration: Example

Example: Approximate the integral $\int_0^1 \sqrt{x+x^2}\,dx$ using Trapezoidal rule with $n = 4$.

Solution:

$[a, b] = [0, 1], f(x) = \sqrt{x+x^2},\ \triangle x = \frac{1-0}{4} = 0.25$

n	x_n	$f(x_n)$	m	$mf(x_n)$
0	0	0	1	0
1	0.25	0.559017	2	1.11803
2	0.5	0.86625	2	1.73205
3	0.75	1.14564	2	2.29129
4	1	1.41421	1	1.41421
				6.55559

$$\int_0^1 \sqrt{x+x^2}\,dx \approx \frac{0-1}{2(4)}[f(0)+2f(0.25)+2f(0.5)+2f(0.75)+f(1)]$$

$$\int_0^1 \sqrt{x+x^2}\,dx \approx 0.819448$$

Numerical Integration:Simpson's Rule

Simpson's Rule: It is used to approximate $\int_a^b f(x)dx$ with a regular partition of the interval $[a, b]$, where $\triangle x = \frac{b-a}{n}$, and n even, by using the formula

$$\int_a^b f(x)dx \approx \frac{b-a}{3n}[f(x_0) + 4f(x_1) + 2f(x_2) +$$

$$+ \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

Numerical Integration: Example

Example: Approximate the integral $\int_0^{10} \sqrt{10x - x^2}\, dx$ using Simpson's rule with $n = 4$.

Solution:

$[a, b] = [0, 10], f(x) = \sqrt{10x - x^2}, \; \triangle x = \frac{10-0}{4} = 2.5$

n	x_n	$f(x_n)$	m	$mf(x_n)$
0	0	0	1	0
1	2.5	4.33013	4	17.3204
2	5	5	2	10
3	7.5	4.33013	4	17.3204
4	10	0	1	0
				44.6408

$$\int_0^{10} \sqrt{10x - x^2}\, dx \approx \frac{10-0}{3(4)}[f(0)+4f(2.5)+2f(5)+4f(7.5)+f(10)]$$

$$\int_0^{10} \sqrt{10x - x^2}\, dx \approx 37.2007$$

Numerical Integration(Exercises)

Exercise 2

1. *Approximate the integral* $\int_2^4 \frac{1}{x-1}dx$ *using Trapezoidal rule with* $n = 4$.
2. *Approximate the integral* $\int_0^2 \frac{x}{x+1}dx$ *using Simpson's rule with* $n = 4$.