INTEGRAL CALCULUS (MATH 106)

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Weekly Objectives

Week 4: Exponential and Logarithmic Functions.

The student is expected to be able to:

1. Find the derivative and integrals natural exponential and logarithmic functions.
2. Find the derivative and integrals general exponential and logarithmic functions.

The natural logarithmic function

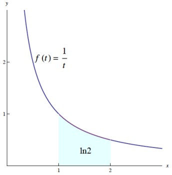
Definition 2.1

For $x > 0$, the natural logarithmic function is defined by

$$\ln x = \int_1^x \frac{1}{t} dt$$

Remark

The domain of the function ln x is the open interval $(0, \infty)$



properties of natural logarithmic function

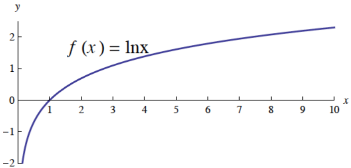
1. If $x > 1$ then $\ln x > 0$
2. $\ln 1 = 0$
3. If $0 < x < 1$ then $\ln x < 0$
4. The range of the function $\ln x$ is $\mathbb{R}$
5. $\lim_{x\to\infty} \ln x = \infty$ and $\lim_{x\to 0+} \ln x = -\infty$
6. $\frac{d}{dx} \ln|x| = \frac{1}{x}$ and $\frac{d}{dx} \ln|f(x)| = \frac{f'(x)}{f(x)}$
7. $\ln|x|$ is the antiderivative of $\frac{1}{x}$
8. $\int \frac{1}{x} dx = \ln|x| + c$ and $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$
9. $\ln(xy) = \ln x + \ln y$, $\ln(\frac{x}{y}) = \ln x - \ln y$ and $\ln x^r = r \ln x$

The natural logarithmic function

The graph of $\ln x$**:**

1. First derivative test :
$$\frac{d}{dx}\ln x = \frac{d}{dx}\int_1^x \frac{1}{t}dt = \frac{1}{x} > 0 \text{ for every } x \in (0, \infty)$$
Hence $\ln x$ is an increasing function on $(0, \infty)$

2. Second derivative test :
$$\frac{d^2}{dx^2}\ln x = \frac{d}{dx}\frac{1}{x} = -\frac{1}{x^2} < 0 \text{ for every } x \in (0, \infty)$$
Hence $\ln x$ is a convex function on $(0, \infty)$



The natural logarithmic function

Basic Rules of Integration :

1. $$\int \tan x \; dx = \ln|\sec x| + c$$
2. $$\int \cot x \; dx = \ln|\sin x| + c$$
3. $$\int \sec x \; dx = \ln|\sec x + \tan x| + c$$
4. $$\int \csc x \; dx = \ln|\csc x - \cot x| + c$$

The natural logarithmic function (Examples)

Example 2.1

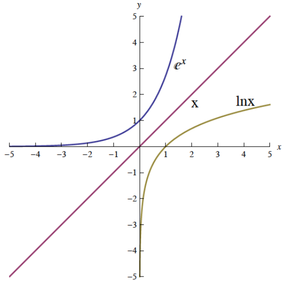
1. $$\int \frac{x^2+2x+3}{x^3+3x^2+9x}\,dx = \frac{1}{3}\int \frac{3x^2+6x+9}{x^3+3x^2+9x}\,dx$$
$$= \frac{1}{3}\ln\left|x^3+3x^2+9x\right| + c$$

2. $$\int \frac{1}{x\sqrt{\ln x}}\,dx = \int (\ln x)^{-\frac{1}{2}}\frac{1}{x}\,dx = \frac{(\ln x)^{\frac{1}{2}}}{\frac{1}{2}} + c$$

The natural exponential function

Definition 3.1

The natural exponential function is the inverse of the natural logarithmic function, and it is denoted by e^x



Properties of the natural exponential function

1. The domain of the function e^x is $\mathbb{R}$
2. The range of the function e^x is the open interval $(0, \infty)$
3. $e^x > 0$ for every $x \in \mathbb{R}$, $e^0 = 1$, and $e \approx 2.71828$ and $\ln e = 1$
4. $\lim_{x \to \infty} e^x = \infty$, $\lim_{x \to -\infty} e^x = 0$
5. $\ln(e^x) = x$ and $e^{\ln x} = x$
6. If $x, y \in \mathbb{R}$, then $e^x e^y = e^{x+y}$, $\frac{e^x}{e^y} = e^{x-y}$, and $(e^x)^y = e^{xy}$
7. $\frac{d}{dx} e^x = e^x$, and $\frac{d}{dx} e^{f(x)} = e^{f(x)} f'(x)$
8. $\int e^x dx = e^x + c$ and $\int e^{f(x)} f'(x) dx = e^{f(x)} + c$

The natural exponential function (Examples)

Example 3.1

1. *Find the value of* x *that satisfies the equation* $\ln \frac{1}{x} = 2$?
$$x = e^{-2} = \frac{1}{e^2}$$
2. *Find* $f'(x)$ *If* $f(x) = e^{5x} + \frac{1}{e^x}$
$f'(x) = 5e^{5x} - e^{-x}$
3. $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}} = 2\int e^{\sqrt{x}} \frac{1}{2\sqrt{x}} = 2e^{\sqrt{x}} + c$
4. $\displaystyle\int_1^e \frac{\sqrt[3]{\ln x}}{x} dx = \int_1^e (\ln x)^{\frac{1}{3}} \frac{1}{x} dx =$
$\displaystyle[\frac{(\ln x)^{\frac{4}{3}}}{\frac{4}{3}}]_1^e = \frac{3}{4}(\ln e)^{\frac{3}{4}} - \frac{3}{4}(\ln 1)^{\frac{4}{3}} = \frac{3}{4}$

The natural exponential function (Exercises)

Exercise 1

1. *Find the value of* x *that satisfies the equation* $e^{5x+3} = 4$?
2. $\displaystyle\int \frac{e^{\sin x}}{\sec x}\,dx =$
3. *Find* $g(x)$ *if* $\displaystyle\int e^{3x^2} g(x)\,dx = -e^{3x^2} + c$

The general exponential function and logarithmic function

Definition 4.1

It has the form a^x where $a > 0$ and $a \neq 1$.
Note: $a^x = e^{x \ln a}$

Derivative and Integration of the general exponential function :

1. $\frac{d}{dx} a^x = a^x \ln a$, and $\int a^x dx = \frac{a^x}{\ln a} + c$
2. $\frac{d}{dx} a^{f(x)} = a^{f(x)} f'(x) \ln a$, and $\int a^{f(x)} f'(x) dx = \frac{a^{f(x)}}{\ln a} + c$

The general exponential function and logarithmic function

Definition 4.2

The general logarithmic function of base a where $a > 0$ and $a \neq 1$ is denoted by $\log_a x$ and it is the inverse function of the general exponential function a^x

Note: $\log_a x = y \Leftrightarrow a^y = x$ and $\log_a x = \frac{\ln x}{\ln a}$
Notations: $\log x = \log_{10} x$ and $\ln x = \log_e x$
Derivative of the general logarithmic function :
$\frac{d}{dx} \log_a |x| = \frac{1}{x} \frac{1}{\ln a}$ and $\frac{d}{dx} \log_a |f(x)| = \frac{f'(x)}{f(x)} \frac{1}{\ln a}$

The general exponential function and logarithmic function (Examples)

1. Find the value of x if $\log_2 x = 3$?
   $\log_2 x = 3 \Leftrightarrow x = 2^3 = 8.$
2. Find y' if $y = (\sin x)^x$
   $y = (\sin x)^x \Rightarrow \ln y = \ln(\sin x)^x = x \ln|\sin x|$
   Differentiate both sides :
   $\frac{y'}{y} = \ln|\sin x| + x\frac{\cos x}{\sin x} = \ln|\sin x| + x \cot x$
3. $$\int x^2 6^{x^3} dx = \frac{1}{3}\int 6^{x^3}(3x^2)\ dx = \frac{6^{x^3}}{3\ln 6} + c$$

The general exponential function and logarithmic function (Exercises)

1. Find $f'(x)$ if $f(x) = (x^2+1)^x$
2. Evaluate $\int \frac{3^{\sqrt{x}}}{\sqrt{x}}$?