INTEGRAL CALCULUS (MATH 106)

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1. The Inverse trigonometric Functions
2. Hyperbolic Function
3. The Inverse Hyperbolic Functions

Weekly Objectives

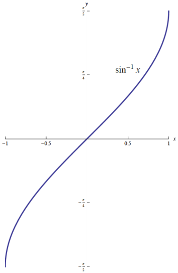
Week 5: The Inverse trigonometric, Hyperbolic and The Inverse Hyoerbolic Functions.

The student is expected to be able to:

1. Find the derivative and integrals The Inverse trigonometric functions.
2. Find the derivative and integrals Hyperbolic functions.
3. Find the derivative and integrals Inverse Hyperbolic functions.

Definition 2.1

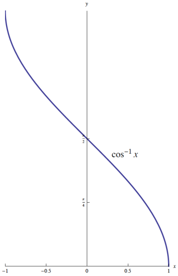
The inverse sine function is denoted by $\sin^{-1}$ *and it is defined as* $y = \sin^{-1} x \Leftrightarrow x = \sin y$, *where* $x \in [-1, 1]$ *and* $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$
The domain of the inverse sine function is $[-1, 1]$
The range of the inverse sine function is $[-\frac{\pi}{2}, \frac{\pi}{2}]$.



Graph of $\sin^{-1} x$

Definition 2.2

The inverse cosine function is denoted by $\cos^{-1}$ *and it is defined as* $y = \cos^{-1} x \Leftrightarrow x = \cos y$, *where* $x \in [-1, 1]$ *and* $y \in [0, \pi]$
The domain of the inverse cosine function is $[-1, 1]$
The range of the inverse cosine function is $[0, \pi]$.



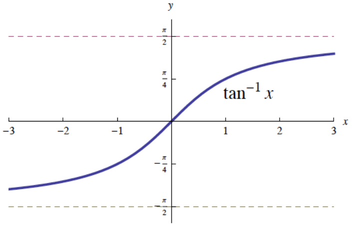
Graph of $\cos^{-1} x$

Definition 2.3

The inverse tangent function is denoted by $\tan^{-1}$ *and it is defined as* $y = \tan^{-1} x \Leftrightarrow x = \tan y$, *where* $x \in \mathbb{R}$ *and* $y \in (-\frac{\pi}{2}, \frac{\pi}{2})$

The domain of the inverse tangent function is $\mathbb{R}$

The range of the inverse tangent function is $(-\frac{\pi}{2}, \frac{\pi}{2})$.



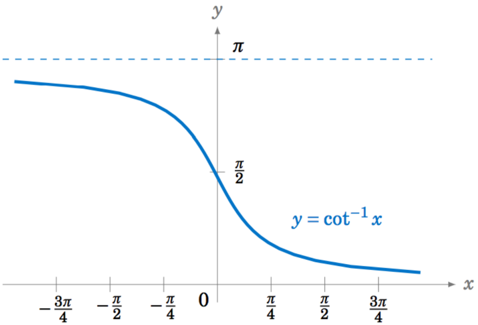
Graph of $\tan^{-1} x$

Definition 2.4

The inverse cotangent function is denoted by $\cot^{-1}$ *and it is defined as* $\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x$, *where* $x \in \mathbb{R}$

The domain of the inverse cotangent function is $\mathbb{R}$

The range of the inverse cotangent function is $(0, \pi)$.

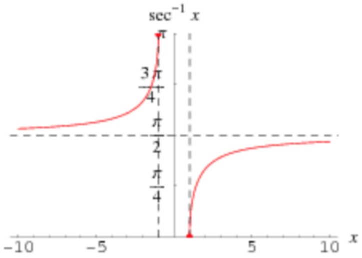


Definition 2.5

The inverse secant function is denoted by $\sec^{-1}$ *and it is defined as* $y = \sec^{-1} x \Leftrightarrow x = \sec y$, *where* $y \in [0, \frac{\pi}{2})$ *if* $x \geq 1$, *and* $y \in [\pi, \frac{3\pi}{2})$ *if* $x \leq -1$

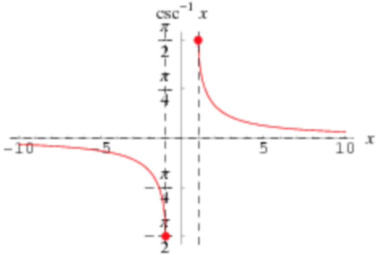
The domain of the inverse secant function is $(-\infty, -1] \cup [1, \infty)$

The range of the inverse secant function is $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$.



Definition 2.6

The inverse cosecant function is denoted by $\csc^{-1}$ *and it is defined as* $\csc^{-1} x = \frac{\pi}{2} - sec^{-1}x$, *where* $|x| \geq 1$
The domain of the inverse cosecant function is $(-\infty, -1] \cup [1, \infty)$
The range of the inverse cosecant function is $[-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$.



Derivatives of the inverse trigonometric functions

1. $\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$, where $|x| < 1$
2. $\frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}$, where $|x| < 1$
3. $\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$
4. $\frac{d}{dx}\cot^{-1}x = \frac{-1}{1+x^2}$
5. $\frac{d}{dx}\sec^{-1}x = \frac{1}{x\sqrt{1-x^2}}$, where $|x| > 1$
6. $\frac{d}{dx}\csc^{-1}x = \frac{-1}{x\sqrt{x^2-1}}$, where $|x| > 1$

Integration of the inverse trigonometric functions

1. $$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}(\frac{x}{a}) + c, \ (|x| < a)$$
   $$\int \frac{f'(x)}{\sqrt{a^2 - [f(x)]^2}} dx = \sin^{-1}(\frac{f(x)}{a}) + c, \ (|f(x)| < a)$$
2. $$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1}(\frac{x}{a}) + c$$
   $$\int \frac{f'(x)}{a^2 + [f(x)]^2} dx = \frac{1}{a} \tan^{-1}(\frac{f(x)}{a}) + c$$
3. $$\int \frac{1}{x\sqrt{x^2 - a^2}} dx = \frac{1}{a} \sec^{-1}(\frac{x}{a}) + c, \ (|x| > a)$$
   $$\int \frac{f'(x)}{f(x)\sqrt{[f(x)]^2 - a^2}} dx = \frac{1}{a} \sec^{-1}(\frac{f(x)}{a}) + c, \ (|f(x)| > a)$$

Integration of the inverse trigonometric functions (Examples)

1. $$\int \frac{x^2}{5+x^6}dx = \frac{1}{3}\int \frac{3x^2}{(\sqrt{5})^2+(x^3)^2}dx = \frac{1}{3}\frac{1}{\sqrt{5}}\tan^{-1}(\frac{x^3}{\sqrt{5}})+c$$
2. $$\int \frac{1}{x\sqrt{1-(\ln x)^2}}dx = \int \frac{(\frac{1}{x})}{\sqrt{(1)^2-(\ln x)^2}}dx = \sin^{-1}(\ln x)+c$$
3. $$\int \frac{1}{\sqrt{e^{2x}-36}}dx = \int \frac{e^x}{e^x\sqrt{(e^x)^2-(6)^2}}dx = \frac{1}{6}\sec^{-1}(\frac{e^x}{6})+c$$

Integration of the inverse trigonometric functions (Exercises)

Exercise 1

Solve the following integrals :

1. $\displaystyle\int \frac{x + \sin^{-1} x}{\sqrt{1 - x^2}} dx$
2. $\displaystyle\int \frac{x + 1}{x^2 + 1} dx$

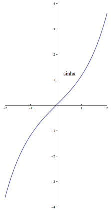
The hyperbolic sine function

Definition 3.1

It is denoted by $\sinh x$ *and it is defined as* $\sinh x = \frac{e^x - e^{-x}}{2}$

Notes:

1. The domain of $\sinh x$ is $\mathbb{R}$ and the range of $\sinh x$ is $\mathbb{R}$.
2. It is an odd function and $\sinh(0) = 0$



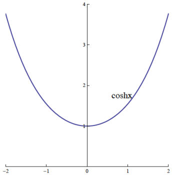
The hyperbolic cosine function

Definition 3.2

It is denoted by $\cosh x$ *and it is defined as* $\cosh x = \frac{e^x + e^{-x}}{2}$

Notes:

1. The domain of $\cosh x$ is $\mathbb{R}$ and the range of $\cosh x$ is $[1, \infty]$.
2. It is an even function and $\cosh(0) = 1$



Definitions :

1. The hyperbolic tangent function is denoted by $\tanh x$ and it is defined as $\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ for every $x \in \mathbb{R}$
2. The hyperbolic cotangent function is denoted by $\coth x$ and it is defined as $\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^x}{e^x - e^{-x}}$ for every $x \in \mathbb{R} - \{0\}$
3. The hyperbolic secant function is denoted by *sech* x and it is defined as *sech* $x = \frac{1}{\cosh x} = \frac{2}{e^x - e^{-x}}$ for every $x \in \mathbb{R}$
4. The hyperbolic cosecant function is denoted by *csch* x and it is defined as *csch* $x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$ for every $x \in \mathbb{R} - \{0\}$

Notes:

1. $cosh^2\ x - sinh^2\ x = 1$ for every $x \in \mathbb{R}$
2. $1 - tanh^2\ x = sech^2\ x$ for every $x \in \mathbb{R}$
3. $coth^2\ x - 1 = csch^2\ x$ for every $x \in \mathbb{R} - \{0\}$

Derivatives of the hyperbolic functions

1. $\frac{d}{dx}\sinh x = \cosh x$, and $\frac{d}{dx}\sinh(f(x)) = \cosh(f(x))f'(x)$
2. $\frac{d}{dx}\cosh x = \sinh x$, and $\frac{d}{dx}\cosh(f(x)) = \sinh(f(x))f'(x)$
3. $\frac{d}{dx} tanh\ x = sech^2\ x$ and $\frac{d}{dx} tanh\ (f(x)) = sech^2\ (f(x))f'(x)$
4. $\frac{d}{dx} coth\ x = -csch^2\ x$ and
   $\frac{d}{dx} coth\ (f(x)) = -csch^2\ (f(x))f'(x)$
5. $\frac{d}{dx} sech\ x = -sech\ x\ tanh\ x$ and
   $\frac{d}{dx} sech\ (f(x)) = -sech\ (f(x))\ tanh\ (f(x))f'(x)$
6. $\frac{d}{dx} csch\ x = -csch\ x\ coth\ x$ and
   $\frac{d}{dx} csch\ (f(x)) = -csch\ (f(x))\ coth\ (f(x))f'(x)$

Integration of the hyperbolic functions

- $\int \sinh x \; dx = \cosh x + c,$
  $\int \sinh(f(x))f'(x)dx = \cosh(f(x)) + c$
- $\int \cosh x \; dx = \sinh x + c,$
  $\int \cosh(f(x))f'(x)dx = \sinh(f(x)) + c$
- $\int sech^2 x \; dx = \tanh x + c$
  $\int sech^2(f(x))f'(x)dx = \tanh(f(x)) + c$
- $\int csch^2 x \; dx = -\coth x + c$
  $\int csch^2(f(x))f'(x)dx = -\coth(f(x)) + c$

Integration of the hyperbolic functions

- $\int sech\ x \tanh x\ dx = -sech\ x + c$

  $\int sech(f(x)) \tanh(f(x)) f'(x) dx = -sech\ (f(x)) + c$

- $\int csch\ x\ coth\ xdx = -csch\ x + c$

  $\int csch\ (f(x))\ coth\ (f(x)) f'(x) dx = -csch\ f(x) + c$

- $\int \tanh x\ dx = \ln|\cosh x| + c$

  $\int \tanh(f(x)) f'(x)\ dx = \ln|\cosh(f(x))| + c$

- $\int \coth x\ dx = \ln|\sinh x| + c$

  $\int \coth(f(x)) f'(x)\ dx = \ln|\sinh(f(x))| + c$

Integration of the hyperbolic functions (Examples)

- $\int x^2 \cosh x^3 \; dx = \frac{1}{3} \int \cosh x^3 \; (3x^2) dx = \frac{1}{3} \sinh x^3 + c$
- $\int (e^x - e^{-x}) sech^2 (e^x + e^{-x}) dx = \tanh(e^x + e^{-x}) + c$
- $\int \frac{\sinh x}{1 + \sinh^2 x} dx = \int \frac{\sinh x}{\cosh^2 x} dx = \int \frac{1}{\cosh x} \frac{\sinh x}{\cosh x} dx$

$$= \int sech \; x \tanh x \; dx = -sech \; x + c$$

- $\int \frac{1}{sech \; x \sqrt{4 - \sinh^2 x}} dx = \int \frac{\cosh x}{\sqrt{(2)^2 - (\sinh x)^2}} dx$

$$= \sin^{-1}(\frac{\sinh x}{2}) + c$$

Definitions

- The inverse hyperbolic sine function is denoted by $\sinh^{-1}$ and it is defined as $y = \sinh^{-1} x \Leftrightarrow x = \sinh y$, where $x \in \mathbb{R}$ and $y \in \mathbb{R}$
- The inverse hyperbolic cosine function is denoted by $\cosh^{-1}$ and it is defined as $y = \cosh^{-1} x \Leftrightarrow x = \cosh y$, where $x \in [1, \infty)$ and $y \in [0, \infty)$
- The inverse hyperbolic tangent function is denoted by $\tanh^{-1}$ and it is defined as $y = \tanh^{-1} x \Leftrightarrow x = \tanh y$, where $x \in [-1, 1]$ and $y \in \mathbb{R}$

Definitions

- The inverse hyperbolic cotangent function is denoted by $\coth^{-1}$ and it is defined as $y = \coth^{-1} x \Leftrightarrow x = \coth y$, where $|x| > 1$ and $y \in \mathbb{R}$.
- The inverse hyperbolic secant function is denoted by $sech^{-1}$ and it is defined as $y = sech^{-1} x \Leftrightarrow x = sech\ y$, where $x \in [0, 1]$ and $y \in [0, \infty)$
- The inverse hyperbolic cosecant function is denoted by $csch^{-1}$ and it is defined as $y = csch^{-1} x \Leftrightarrow x = csch\ y$, where $x \in \mathbb{R}$ and $y \in \mathbb{R} - \{0\}$

Derivatives of the inverse hyperbolic functions

- $\frac{d}{dx}\sinh^{-1}x = \frac{1}{\sqrt{1+x^2}}$,
  $\frac{d}{dx}\sinh^{-1}f(x) = \frac{f'(x)}{\sqrt{1+f((x))^2}}$.
- $\frac{d}{dx}\cosh^{-1}x = \frac{1}{\sqrt{x^2-1}}$, where $x > 1$
  $\frac{d}{dx}\cosh^{-1}f(x) = \frac{f'(x)}{\sqrt{(f(x))^2-1}}$, where $|f(x)| > 1$
- $\frac{d}{dx}\tanh^{-1}x = \frac{1}{1-x^2}$, where $|x| > 1$
  $\frac{d}{dx}\tanh^{-1}f(x) = \frac{f'(x)}{1-(f(x))^2}$, where$|f(x)| > 1$

Derivatives of the inverse hyperbolic functions

- $\frac{d}{dx} coth^{-1}x = \frac{-1}{1-x^2}$ where $|x| > 1$
  $\frac{d}{dx} coth^{-1}f(x) = \frac{-f'(x)}{1-(f(x))^2}$ where $|f(x)| > 1$
- $\frac{d}{dx} sech^{-1}x = \frac{-1}{x\sqrt{1-x^2}}$ where $0 < x < 1$
  $\frac{d}{dx} sech^{-1}f(x) = \frac{-f'(x)}{f(x)\sqrt{1-(f(x))^2}}$ where $0 < f(x) < 1$
- $\frac{d}{dx} csch^{-1}x = \frac{-1}{|x|\sqrt{1+x^2}}$, where $x \neq 0$
  $\frac{d}{dx} csch^{-1}f(x) = \frac{-f'(x)}{|f(x)|\sqrt{1+(f(x))^2}}$, where $f(x) \neq 0$

Derivatives of the inverse hyperbolic functions (Examples)

1. Find $f'(x)$ if $f(x) = \tanh^{-1} 3x$?
   $f'(x) = \frac{3}{1-(3x)^2} = \frac{3}{1-9x^2}$
2. Find $f'(x)$ if $f(x) = \sinh^{-1} \sqrt{x}$?
   $f'(x) = \frac{\frac{1}{2\sqrt{x}}}{\sqrt{1+(\sqrt{x})^2}} = \frac{1}{2\sqrt{x}\sqrt{1+x}}$
3. Find $f'(x)$ if $f(x) = sech^{-1}(\cos 2x)$?
   $f'(x) = \frac{-(-2\sin 2x)}{\cos 2x\sqrt{1-(\cos 2x)^2}} = \frac{2\sin 2x}{\cos 2x\sqrt{1-\cos^2 2x}}$

Integration of the inverse hyperbolic functions

- $$\int \frac{1}{\sqrt{a^2+x^2}}dx = \sinh^{-1}(\frac{x}{a}) + c$$
  $$\int \frac{f'(x)}{\sqrt{a^2+(f(x))^2}}dx = \sinh^{-1}(\frac{f(x)}{a}) + c$$
- $$\int \frac{1}{\sqrt{x^2-a^2}}dx = cosh^{-1}(\frac{x}{a}) + c, (x > a)$$
  $$\int \frac{f'(x)}{\sqrt{(f(x))^2-a^2}}dx = cosh^{-1}(\frac{f(x)}{a}) + c,\ (f(x) > a)$$
- $$\int \frac{1}{a^2-x^2}dx = \frac{1}{a}tanh^{-1}(\frac{x}{a}) + c(|x| < a)$$
  $$\int \frac{f'(x)}{a^2-(f(x))^2}dx = \frac{1}{a}tanh^{-1}(\frac{f(x)}{a}) + c,\ (|f(x)| < a)$$

Integration of the inverse hyperbolic functions

- $$\int \frac{1}{x\sqrt{a^2 - x^2}} dx = -\frac{1}{a} sech^{-1}(\frac{x}{a}) + c, \ (0 < x < a)$$
  $$\int \frac{f'(x)}{f(x)\sqrt{a^2 - (f(x))^2}} dx = -\frac{1}{a} sech^{-1}(\frac{f(x)}{a}) + c,$$
  $(0 < f(x) < a)$
- $$\int \frac{1}{x\sqrt{x^2 + a^2}} dx = -\frac{1}{a} csch^{-1}(\frac{x}{a}) + c, \ (x \neq 0)$$
  $$\int \frac{f'(x)}{x\sqrt{(f(x))^2 + a^2}} dx = -\frac{1}{a} csch^{-1}(\frac{f(x)}{a}) + c, \ (f(x) \neq 0)$$

Integration of the inverse hyperbolic functions

1. $$\int \frac{e^x}{1-e^{2x}}dx = \int \frac{e^x}{(1)^2-(e^x)^2}dx = \tanh^{-1}(e^x)+c$$

2. $$\int \frac{1}{\sqrt{x}\sqrt{4+x}}dx = 2\int \frac{\frac{1}{2\sqrt{x}}}{\sqrt{(2)^2+(\sqrt{x})^2}}dx$$
$$= 2\sinh^{-1}(\frac{\sqrt{x}}{2})+c$$

3. $$\int \frac{1}{\sqrt{1+e^{2x}}}dx = \int \frac{e^x}{e^x\sqrt{1+e^{2x}}}dx = -csch^{-1}(e^x)+c$$