INTEGRAL CALCULUS (MATH 106)

Dr.Maamoun TURKAWI

king saud university

October 4, 2020

1. Indeterminate Forms and l'Hopital's Rule.

2. Integration By Parts

Weekly Objectives

Week 6: Indeterminate Forms and l'Hopital's Rule and integration by parts.

The student is expected to be able to:

1. handles with Indeterminate Forms and uses Hopital's Rule.
2. integrate the functions using integration by parts.

Indeterminate Forms

Theorem (L'Hopital's Rule)

Suppose that f and g are differentiable on the interval (a, b), except possibly at a point $c \in (a, b)$ and that $g'(x) \neq 0$ on (a, b), except possibly at c. Suppose further that $\lim_{x \to c} \frac{f(x)}{g(x)}$ has the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and that $\lim_{x \to c} \frac{f'(x)}{g'(x)} = L(\text{or } \pm\infty)$.

Then, $\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$.

Remark

The conclusion of the theorem also holds if $\lim_{x \to c} \frac{f(x)}{g(x)}$ is replaced with $\lim_{x \to c^-} \frac{f(x)}{g(x)}$, $\lim_{x \to c^+} \frac{f(x)}{g(x)}$, $\lim_{x \to -\infty} \frac{f(x)}{g(x)}$ or $\lim_{x \to +\infty} \frac{f(x)}{g(x)}$. (In each case, we must make appropriate adjustment of the hypothesis.)

Types of indeterminate forms:

1. $\frac{0}{0}$ or $\frac{\infty}{\infty}$
2. $\infty - \infty$ or $-\infty + \infty$
3. $0.\infty$ or $0(-\infty)$
4. $0^0, 1^\infty, 1^{-\infty}$ or ∞^0

Example 2.1

$$\lim_{x\to 1} \frac{\sqrt{x}}{\ln x} = \frac{0}{0}$$

Apply L'Hopital's rule

$$\lim_{x\to 1} \frac{\sqrt{x}}{\ln x} = \lim_{x\to 1} \frac{(\frac{1}{2\sqrt{x}})}{(\frac{1}{x})} = \lim_{x\to 1} \frac{x}{2\sqrt{x}} = \frac{1}{2}$$

Example 2.2

1. $$\lim_{x\to(\frac{\pi}{2})^-} \frac{2-\sec x}{3\tan x} = \frac{-\infty}{\infty}$$
   Apply L'Hopital's rule
   $$\lim_{x\to(\frac{\pi}{2})^-} \frac{2-\sec x}{3\tan x} = \lim_{x\to(\frac{\pi}{2})^-} \frac{-\sec x\tan x}{3\sec^2 x} = \lim_{x\to(\frac{\pi}{2})^-} \frac{-\tan x}{3\sec x} =$$
   $$\lim_{x\to(\frac{\pi}{2})^-} \frac{-\sin x}{3} = -\frac{1}{3}$$
2. $$\lim_{x\to1^+} \left(\frac{3}{\ln x} - \frac{2}{x-1}\right) = (\infty - \infty)$$
   $$\lim_{x\to1^+} \frac{3(x-1)-2\ln x}{(x-1)\ln x} = \frac{0}{0}$$
   Apply L'Hopital's rule
   $$\lim_{x\to1^+} \frac{3(x-1)-2\ln x}{(x-1)\ln x} = \lim_{x\to1^+} \frac{3-\frac{2}{x}}{\ln x+(x-1)\frac{1}{x}} = \lim_{x\to1^+} \frac{3-\frac{2}{x}}{\ln x+1-\frac{1}{x}} = \infty$$

Integration By Parts

It is used to solve integration of a product of two functions using the formula:

$$\int u\ dv = uv - \int v\ du$$

1. $\int xe^x dx$, We put, $u = x \quad dv = e^x\ dx$, Then $du = dx \quad v = e^x$
$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + c$$
2. $\int_0^\pi x \sin x\ dx$, We put $u = x \quad dv = \sin x\ dx$, Then,
$du = dx \quad v = -\cos x$
$$\int_0^\pi x \sin x\ dx = [-x \cos x]_0^\pi + \int_0^\pi \cos x dx = [-x \cos x]_0^\pi + [\sin x]_0^\pi$$
$$[(-\pi \cos \pi) - (-(0) \cos 0)] + [\sin \pi - \sin 0] = \pi$$

Integration By Parts: Examples

- $$\int xe^x \, dx = (x-1)e^x + c$$
  $$\int x^2e^x \, dx = (x^2-2x+2)e^x + c$$
  $$\int x^3e^x \, dx = (x^3-3x^2+6x-6)e^x + c$$

Integration By Parts: Examples

- $\displaystyle\int x\cos x\ dx = x\sin x + \cos x + c$

  $\displaystyle\int x^2\cos x\ dx = (x^2 - 2)\sin x + 2x\cos x + c$

  $\displaystyle\int x^3\cos x\ dx = (x^3 - 6x)\sin x + (3x^2 - 6)\cos x + c$

  $\displaystyle\int x^4\cos x\ dx = (x^4 - 12x^2 + 24)\sin x + (4x^3 - 24x)\cos x + c$

Integration By Parts: Examples

- $\int x \sin x \; dx = -x \cos x + \sin x + c$

  $\int x^2 \sin x \; dx = (-x^2 + 2) \cos x + 2x \sin x + c$

  $\int x^3 \sin x \; dx = (-x^3 + 6x) \cos x + (3x^2 - 6) \sin x + c$

  $\int x^4 \sin x \; dx = (-x^4 + 12x^2 - 24) \cos x + (4x^3 - 24x) \sin x + c$

Integration By Parts: Examples

Evaluate $\int \cos(\ln(x))\ dx$.

Letting: $u = \ln(x)$, we have $du = 1/x\ dx$.

$$du = \frac{1}{x}\ dx \Rightarrow x \cdot du = dx.$$

Since $u = \ln(x)$, we can use inverse functions and conclude that $e^{\ln(x)} = e^u \Rightarrow x = e^u$. therefore we have that $dx = x \cdot du = e^u\ du$.

$$\begin{aligned}\int \cos(\ln(x))\ dx &= \int e^u \cos(u)\ du \\ &= \frac{1}{2}e^u\left(\sin(u) + \cos(u)\right) + C \\ &= \frac{1}{2}e^{\ln(x)}\left(\sin(\ln(x)) + \cos(\ln(x))\right) + C \\ &= \frac{1}{2}x\left(\sin(\ln(x)) + \cos(\ln(x))\right) + C.\end{aligned}$$