INTEGRAL CALCULUS (MATH 106)

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October 11, 2020

Weekly Objectives

Week 7: Trigonometrics Integrals and substutions.

The student is expected to be able to:

1. handles with Integrals Involving Trigonometric Functions.
2. Applying Trigonometric substitution to integrals.

Integrals Involving Trigonometric Functions

First : Integrals of the forms

$$\int \sin ax \cos bx \; dx, \quad \int \sin ax \sin bx \; dx, \quad \int \cos ax \cos bx \; dx$$

Wher $a, b \in \mathbb{Z}$

1. The integral $\int \sin ax \cos bx \; dx$ can be solved using the formula
   $\sin ax \cos bx = \frac{1}{2}[\sin(ax + bx) + \sin(ax - bx)]$
2. The integral $\int \sin ax \sin bx \; dx$ can be solved using the formula
   $\sin ax \sin bx = \frac{1}{2}[\cos(ax - bx) - \cos(ax + bx)]$
3. The integral $\int \cos ax \cos bx \; dx$ can be solved using the formula
   $\cos ax \cos bx = \frac{1}{2}[\cos(ax + bx) + \cos(ax - bx)]$

Integrals Involving Trigonometric Functions (Examples)

1. $$\int \sin 3x \cos 2x \; dx = \frac{1}{2}\int [\sin 5x + \sin x] dx = \frac{1}{2}\int \sin 5x \; dx + \frac{1}{2}\int \sin x \; dx = -\frac{1}{10}\cos 5x - \frac{1}{2}\cos x + c$$
2. $$\int \sin x \sin 3x \; dx = \frac{1}{2}\int [\cos 2x - \cos 4x] dx = \frac{1}{2}\int \cos 2x \; dx - \frac{1}{2}\int \cos 4x \; dx = \frac{1}{4}\sin 2x - \frac{1}{8}\sin 4x + c$$
3. $$\int \cos 5x \cos 2x \; dx = \frac{1}{2}\int [\cos 7x + \cos 3x] dx = \frac{1}{2}\int \cos 7x \; dx + \int \cos 3x \; dx = \frac{1}{4}\sin 7x + \frac{1}{6}\sin 3x + c$$

Integrals Involving Trigonometric Functions

Second : Integrals of the forms

$$\int \sin^n x \cos^m x \ dx, \quad \int \sinh^n x \cosh^m x \ dx, \qquad Where\ n, m \in \mathbb{N}$$

The above two integrals can be solved by substitution if n or m is odd.

1. If n is odd: The substitution $u = \cos x$ can be used to solve $\int \sin^n x \cos^m x \ dx$
   The substitution $u = \cosh x$ can be used to solve $\int \sinh^n x \cosh^m x \ dx$
2. If m is odd: The substitution $u = \sin x$ can be used to solve $\int \sin^n x \cos^m x \ dx$
   The substitution $u = \sinh x$ can be used to solve $\int \sinh^n x \cosh^m x \ dx$

Integrals Involving Trigonometric Functions (Examples)

Example 2.1

Evaluate $I = \int \sin^5 x \cos^4 x dx$

$$\int \sin^5 x \cos^4 x dx = \int (\sin^2 x)^2 \cos^4 x \sin x dx$$
$$= \int (1 - \cos^2 x) \cos^4 x \sin x \; dx$$

to solve this integral put

$$u = \cos x \Rightarrow -du = \sin x \; dx$$

$$I = -\int (1-u^2)^2 u^4 du = -\int u^4 - 2u^6 + u^8 du$$
$$= -[\frac{u^5}{5} - \frac{2u^7}{7} + \frac{u^9}{9}] + c = -[\frac{\cos^5 x}{5} - \frac{2\cos^7 x}{7} + \frac{\cos^9 x}{9}] + c$$

Integrals Involving Trigonometric Functions (Examples)

Example 2.2

Evaluate $I = \int \sin^7 \cos^3 x \; dx$

$\int \sin^7 \cos^3 x \; dx = \int \sin^7 x(1 - \sin^2 x) \cos x \; dx$

to solve this integral put

$$u = \sin x \Rightarrow du = \cos x \; dx$$

$$I = \int u^7(1 - u^2) \; du = \int u^7 - u^9 \; du = \frac{u^8}{8} - \frac{u^{10}}{10} + c$$

$$= \frac{\sin^8 x}{8} - \frac{sin^{10}}{10} + c$$

Integrals Involving Trigonometric Functions (Examples)

- $\int \sinh^3 x \cosh^2 x \; dx$ to solve this integral put
  $u = \cosh x \Rightarrow du = \sinh x$
  $\int \sinh^3 x \cosh^2 x \; dx = \int (\cosh^2 x - 1) \cosh^2 x \sinh x \; dx =$
  $\int (u^2 - 1)u^2 du = \int (u^4 - u^2) du = \frac{u^5}{5} - \frac{u^3}{3} + c$

$$= \frac{\cosh^5 x}{5} - \frac{\cosh^3 x}{3} + c$$

Special cases :

1. $\int \sin^2 x \; dx = \frac{1}{2} \int [1 - \cos 2x] \; dx = \frac{1}{2}[x - \frac{\sin 2x}{2}] + c$
2. $\int \cos^2 x \; dx = \frac{1}{2} \int [1 + \cos 2x] \; dx = \frac{1}{2}[x + \frac{\sin 2x}{2}] + c$

Integrals Involving Trigonometric Functions

Third : Integrals of the forms

$$\int \sec^n x \tan^m x \; dx, \quad \int csc^n x \cot^m x \; dx,$$

$$\int sech^n x \tanh^m x \; dx, \quad \int csch^n x \coth^m x \; dx$$

The above four integrals can be solved by substitution if n is even or m is odd.

Integrals Involving Trigonometric Functions

1. If n is even:
   The substitution $u = \tan x$ can be used to solve $\int \sec^n x \tan^m x \; dx$.
   The substitution $u = \cot x, \;\; u = \tanh x$ and $u = \coth x$ can be used to solve the other three integrals respectively.
2. If m is odd:
   The substitution $u = \sec x$ can be used to solve $\int \sec^n x \tan^m x \; dx$.
   The substitutions $u = csc \;\; x, u = sech \;\; x$ and $u = csch \;\; x$ can be used to solve the other theree integrals respictively.

Integrals Involving Trigonometric Functions (Examples)

Example 2.3

Evaluate $I = \int \tan^3 x \sec^3 x \; dx$

to solve this integral put: $u = \sec x \Rightarrow du = \sec x \tan x \; dx$

$I = \int \tan^3 x \sec^3 x \; dx = \int (\sec^2 x - 1) \sec^2 x \sec x \tan x \; dx$

$$= (u^2 - 1)u^2 \; du = \int u^4 - u^2 \; du$$

$$= \frac{u^5}{5} - \frac{u^3}{3} + c$$

$$= \frac{\sec^5}{5} - \frac{\sec^3}{3} + c$$

Integrals Involving Trigonometric Functions (Examples)

Example 2.4

Evaluate $I = \int tanh^3 x \ sech \ x \ dx$

to solve this integral put: $u = sech \ x \Rightarrow -du = sech \ x \tanh x \ dx$

$I = \int tanh^3 x \ sech \ x \ dx = \int (1 - sech^2 x) sech \ x \ tanh \ x \ dx$

$$= -\int (1 - u^2) du = -u + \frac{u^3}{3} + c$$

$$= -sech \ x + \frac{sech^3 x}{3} + c$$

Trigonometric Substitutions

If the integrand contains a term of the form
$\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$ or $\sqrt{x^2 - a^2}$ where $a > 0$,
then trigonometric substitutions can be used to solve the integral.

1. An integral involving $\sqrt{a^2 - x^2}$ use the substitution $x = a\sin\theta$ where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ to solve the integral.
2. An integral involving $\sqrt{a^2 + x^2}$ use the substitution $x = a\tan\theta$ where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ to solve the integral.
3. An integral involving $\sqrt{x^2 - a^2}$ use the substitution $x = a\sec\theta$ where $0 \leq \theta < \frac{\pi}{2}$ to solve the integral.

Trigonometric Substitutions (Examples)

Solve the following integral: $\displaystyle\int \frac{1}{x^2\sqrt{16-x^2}}dx$

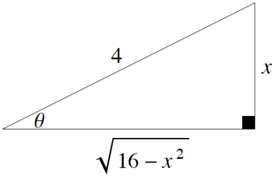
$$\int \frac{1}{x^2\sqrt{16-x^2}}dx = \int \frac{1}{x^2\sqrt{(4)^2-x^2}}dx,$$

Put $x = 4\sin\theta \Rightarrow dx = 4\cos\theta d\theta$

$$I = \int \frac{4\cos\theta}{16\sin^2\theta\sqrt{16-16\sin\theta}}d\theta$$

$$= \int \frac{4\cos\theta}{16\sin^2\theta\ 4\cos\theta}d\theta$$

$$= \frac{1}{16}\int \frac{1}{\sin^2\theta}d\theta = \frac{1}{16}\int \csc^2\theta d\theta = \frac{1}{16}\cot\theta + c$$



$$\int \frac{1}{x^2\sqrt{16-x^2}}dx = -\frac{1}{16}\frac{\sqrt{16-x^2}}{x} + c$$

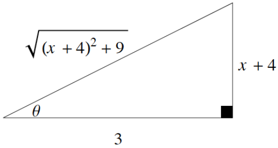
Trigonometric Substitutions (Examples)

Solve the following integral:

$$\int \frac{1}{[x^2+8x+25]^{\frac{3}{2}}}\ dx$$

$$I=\int \frac{1}{[(x+4)^2+3^2]^{\frac{3}{2}}}\ dx.$$

Put $x+4=3\tan\theta \Rightarrow dx=3\sec^2\theta\ d\theta$



$$\int \frac{1}{[x^2+8x+25]^{\frac{3}{2}}}\ dx=\int \frac{3\sec^2\theta}{(9\tan^2\theta+9)^{\frac{3}{2}}}\ d\theta$$

$$=\int \frac{3\sec^{\theta}}{(9\sec^2\theta)^{\frac{3}{2}}}\ d\theta=\int \frac{3\sec^2\theta}{27\sec^3\theta}d\theta$$

$$=\frac{1}{9}\int \frac{1}{\sec\theta}d\theta=\frac{1}{9}\sin\theta+c$$

$$\int \frac{1}{[x^2+8x+25]^{\frac{3}{2}}}\ dx=\frac{1}{9}\frac{x+4}{\sqrt{x^2+8x+25}}+c$$

Trigonometric Substitutions (Examples)

Solve the following integral: $\displaystyle\int \frac{\sqrt{x^2-4}}{x^2}\,dx$

Put $x = 2\sec\theta \Rightarrow dx = 2\sec\theta\tan\theta\ d\theta$

$$\int \frac{\sqrt{x^2-4}}{x^2}\,dx = \int \frac{\sqrt{4\sec^2\theta-4}\ 2\sec\theta\tan\theta}{4\sec^2\theta}d\theta$$

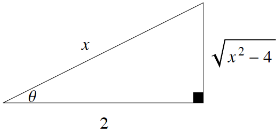
$$= \int \frac{(2\tan\theta)(2\sec\theta\tan\theta)}{4\sec^2\theta}d\theta$$

$$= \int \frac{\tan^2\theta}{\sec\theta}d\theta = \int \frac{\sec^2\theta-1}{\sec\theta}d\theta$$

$$= \int \frac{\sec^2\theta}{\sec\theta}d\theta - \int \frac{1}{\sec\theta}d\theta$$

$$= \int \sec\theta d\theta - \int \cos\theta d\theta$$

$$= \ln|\sec\theta+\tan\theta| - \sin\theta + c$$



$$\int \frac{\sqrt{x^2-4}}{x^2}\,dx = \ln\left|\frac{x}{2}+\frac{\sqrt{x^2-4}}{2}\right| - \frac{\sqrt{x^2-4}}{x} + c$$