INTEGRAL CALCULUS (MATH 106)

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1. Integration of Rational Function

Weekly Objectives

Week 8: Integration of Rational Function.

The student is expected to be able to:

1. solve integrals of rational functions (Partial fractions).

Method of Partial fractions

Definition 2.1

Linear Factor:

A linear factor is a polynomial of degree 1. It has the form $ax + b$ *where* $a, b \in \mathbb{R}$ *and* $a \neq 0$.
Such $x, 3x,$ *and* $2x - 7$

Definition 2.2

Irreducible Quadratic:

An irreducible quadratic is a polynomial of degree 2. It has the form $ax^2 + bx + c$, *where* $a, b, c \in \mathbb{R}$ *and* $b^2 - 4ac < 0$.
Such $x^2 + 9$ *and* $x^2 + x + 1$.

Method of Partial fractions

What is the Partial Fraction?

It is re-expressing a rational function (a ratio of polynomial function) as a sum of simpler fraction.

Let $h(x) = \frac{P(x)}{Q(x)} = \dfrac{a_0 x^n + a_1 x^{n-1} + \cdots + a - n - 1x + a_n}{b_o x^m + b_1 x^{m-1} + \cdots + b_{m-1} x + b_m}$ be a rational function, where $P(x)$, $Q(x)$ are polynomials, we have two cases:

1. degree $P(x) <$ degree $Q(x)$ use method of partial fractions.
2. degree $P(x) \geq$ degree $Q(x)$ use long division of polynomials, then use method of partial fractions.

Method of Partial fractions

How do we create partial functions?

1. If we can write $Q(x)$ as a linear factors
$b_o x^m + b_1 x^{m-1} + \cdots + b_{m-1} x + b_m = (x-a)^m, \; a \in \mathbb{R}, \; m \in \mathbb{N}$
Then: $h(x) = \dfrac{A_0}{(x-a)^m} + \dfrac{A_1}{(x-a)^{m-1}} + \cdots + \dfrac{A_{m-1}}{x-a}, m \in \mathbb{N}$
2. If we can write $Q(x)$ as a irreducible quadratic factors
$b_o x^m + b_1 x^{m-1} + \cdots + b_{m-1} x + b_m = (ax^2+bx+c)^n, \; a, b, c \in \mathbb{N}$
and $b^2 - 4ac < 0$
Then: $h(x) =$
$$\frac{B_0 x + C_0}{(ax^2+bx+c)^n} + \frac{B_1 x + C_1}{(ax^2+bx+c)^{n-1}} + \cdots + \frac{B_{n-1} x + C_{n-1}}{ax^2+bx+c}.$$

Method of Partial fractions

Some time we can write $Q(x)$ as a product of linear factors and irreducible quadratics.

Then

$$h(x) = \frac{A_0}{(x-a)^m} + \frac{A_1}{(x-a)^{m-1}} + \cdots + \frac{A_{m-1}}{x-a}$$

$$+\frac{B_0x + C_0}{(ax^2+bx+c)^n} + \frac{B_1x + C_1}{(ax^2+bx+c)^{n-1}} + \cdots + \frac{B_{n-1}x + C_{n-1}}{ax^2+bx+c}.$$

Method of Partial fractions

Example 2.1

Determine the partial fraction for: $\dfrac{x-3}{x^2-4}$

$$\frac{x-3}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2} \Rightarrow x-3 = A(x+2) + B(x-2)$$

$x = -2 \Rightarrow -5 = -4B \Rightarrow B = \frac{5}{4}$

$x = 2 \Rightarrow -1 = 4A \Rightarrow A = \frac{-1}{4}$

So: $\dfrac{x-3}{(x-2)(x+2)} = \dfrac{-1}{4(x-2)} + \dfrac{5}{4(x+2)}$

Method of Partial fractions

Now Integrate:

Determine $\displaystyle\int \frac{x-3}{x^2-4}dx$

$$\int \frac{x-3}{x^2-4}dx = \int \left[\frac{-1}{4(x-2)} + \frac{5}{4(x+2)}\right]dx$$

$$= -\frac{-1}{4}\ln\left|x-2\right| + \frac{5}{4}\ln\left|x+2\right| + c$$

Method of Partial fractions

Example 2.2

Determine $\int \frac{x-3}{x^2+4x}dx$

Note that degree $P(x) <$ *degree* $Q(x)$

$$\frac{x-3}{x^2+4x} = \frac{x-3}{x(x+4)} = \frac{A}{x} + \frac{B}{x+4} \Rightarrow x-3 = A(x+4) + Bx$$

$x = -4 \Rightarrow -7 = -4B \Rightarrow B = \frac{7}{4}$

$x = 0 \Rightarrow -3 = 4A \Rightarrow A = \frac{-3}{4}$

Method of Partial fractions

Now we can write:

$$\frac{x-3}{x^2-4x} = \frac{-3}{4x} + \frac{7}{4(x+4)}.$$

$$\int \frac{x-3}{x^2-4x}dx = \int \frac{-3}{4x}dx + \int \frac{7}{4(x+4)}dx$$

$$= -\frac{3}{4}\ln|x| + \frac{7}{4}\ln|x+4| + C$$

$$= \frac{\ln|x+4|^{\frac{7}{4}}}{\ln|x|^{\frac{3}{4}}} + C$$

Method of Partial fractions

Example 2.3

Determine $\int \frac{x^2-2}{x^3+3x^2+2x}dx$

Note that degree $P(x) <$ *degree* $Q(x)$

$$\frac{x^2-2}{x^3+3x^2+2x} = \frac{x^2-2}{x(x+2)(x+1)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x+1}$$

$$\Rightarrow x^2-2 = A(x+2)(x+1) + Bx(x+1) + Cx(x+2)$$

$$x = 0 \Rightarrow -2 = 2A \Rightarrow A = -1$$

$$x = -2 \Rightarrow 2 = 2B \Rightarrow b = 1$$

$$x = -1 \Rightarrow -1 = -C \Rightarrow c = 1$$

Method of Partial fractions

Now we can write:

$$\frac{x^2-2}{x^3+3x^2+2x} = \frac{-1}{x} + \frac{1}{x+2} + \frac{1}{x+1}.$$

$$\int \frac{x^2-2}{x^3+3x^2+2x}dx = \int \frac{-1}{x}dx + \int \frac{1}{x+2}dx + \int \frac{1}{x+1}$$

$$= -\ln|x| + \ln|x+2| + \ln|x+1| + c$$

Method of Partial fractions

Example 2.4

Determine $\int \frac{x^3 - 2x^2 + x + 1}{x^4 + 5x^2 + 4} dx$

Note that degree $P(x) <$ degree $Q(x)$.

We can write: $x^4 + 5x^2 + 4 = (x^2 + 4)(x^2 + 1)$

$$\frac{x^3 - 2x^2 + x + 1}{x^4 + 5x^2 + 4} = \frac{Ax + b}{x^2 + 4} + \frac{Cx + D}{x^2 + 1}$$

$$\Rightarrow x^3 - 2x^2 + x + 1 = (Ax + B)(x^2 + 1) + (Cx + D)(x^2 + 4)$$
$$= (A + C)x^3 + (B + D)x^2 + (A + 4C)x + (B + 4D)$$

$A = 1, \quad B = -3, \quad C = 0, \quad D = 1$

Method of Partial fractions

Now we can write:

$$\frac{x^3 - 2x^2 + x + 1}{x^4 + 5x^2 + 4} = \frac{x-3}{x^2+4} + \frac{1}{x^2+1}$$

$$\int \frac{x^3 - 2x^2 + x + 1}{x^4 + 5x^2 + 4} dx = \int [\frac{x-3}{x^2+4} + \frac{1}{x^2+1}] dx$$
$$= \frac{1}{2} \ln |x^2 + 4| - \frac{3}{2} \tan^{-1} \frac{x}{2} + \tan^{-1} x + c$$

Method of Partial fractions

Example 2.5

Determine $\int \frac{x^2+3}{x^2-x-2}dx$

Note that degree $P(x) \geq$ *degree* $Q(x)$

Here Divide First

$$\frac{x^2+3}{x^2-x-2} = 1 + \frac{x+5}{x^2-x-2}$$

$$\frac{x+5}{x^2-x-2} = \frac{x+5}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1} \Rightarrow$$

$$x+5 = A(x+1) + B(x-2) x = 2 \Rightarrow 7 = 3A \Rightarrow A = \frac{7}{3}$$

$$x = -1 \Rightarrow 4 = -3B \Rightarrow B = \frac{-4}{3}$$

Method of Partial fractions

Now we can write:

$$\frac{x^2+3}{x^2-x-2} = 1 + \frac{7}{3(x-2)} - \frac{4}{3(x+1)}$$

$$\int \frac{x^2+3}{x^2-x-2}dx = \int \left[1 + \frac{7}{3(x-2)} - \frac{4}{3(x+1)}\right]dx$$

$$= x + \frac{7}{3}\ln\left|x-2\right| - \frac{4}{3}\ln\left|x+1\right| + c$$

Method of Partial fractions

Example 2.6

Determine $I = \displaystyle\int \frac{x^4+1}{(x+1)(x^2+x+1)}dx$

Note that degree $P(x) \geq$ *degree* $Q(x)$

$$\frac{x^4+1}{(x+1)(x^2+x+1)} = (x-2) + \frac{2x^2+3x+3}{(x+1)(x^2+x+1)}$$

$$I = \underbrace{\int (x-2)\ dx}_{I_1} + \underbrace{\int \frac{2x^2+3x+3}{(x+1)(x^2+x+1)}dx}_{I_2}$$

Method of Partial fractions

Now for I_2 we have de to applied method of partial fraction

$$\frac{2x^2+3x+3}{(x+1)(x^2+x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+x+1},$$

$$\begin{aligned} \Rightarrow 2x^2+3x+3 &= A(x^2+x+1)+(Bx+C)(x+1) \\ &= Ax^2+Ax+A+Bx^2+Bx+Cx+C \\ &= (A+B)x^2+(A+B+C)x+(A+C) \end{aligned}$$

$A+B=2$

$A+B+C=3$

$A+C=3$

So: $A=2$, $B=0$, and $C=1$.

Method of Partial fractions

Now we can write:

$$\frac{2x^2+3x+3}{(x+1)(x^2+x+1)} = \frac{2}{x+1} + \frac{1}{x^2+x+1}$$

$$I = I_1 + I_2 = \int (x-2)\ dx + \int \frac{2x^2+3x+3}{(x+1)(x^2+x+1)} dx$$

$$= \underbrace{\int (x-2)\ dx}_{I_1} + \underbrace{\underbrace{\int \frac{2}{x+1} dx}_{J_1} + \underbrace{\int \frac{1}{x^2+x+1} dx}_{J_2}}_{I_2}$$

Method of Partial fractions

$$J_2 = \int \frac{1}{x^2+x+1}dx = \int \frac{1}{x^2+x+\frac{1}{4}+\frac{3}{4}}dx$$
$$= \int \frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}dx = \frac{2}{\sqrt{3}}\tan^{-1}(\frac{2x+1}{\sqrt{3}}) + c$$

So,

$$\int \frac{x^4+1}{(x+1)(x^2+x+1)}dx = I_1 + \underbrace{J_1 + J_2}_{I_2}$$
$$= \underbrace{x^2 - 2x}_{I_1} + \underbrace{2\ln|x+1|}_{J_1} + \underbrace{\frac{2}{\sqrt{3}}\tan^{-1}(\frac{2x+1}{\sqrt{3}})}_{J_2} + c$$

Exercises

1. $$\int \frac{6x+7}{(x+2)^2}dx$$
2. $$\int \frac{-2x+4}{(x^2+1)(x-1)^2}dx$$
3. $$\int \frac{x}{x^2+2x-3}dx$$
4. $$\int \frac{x^2}{(x-1)^2(x+1)}dx$$
5. $$\int \frac{x^3-5x+7}{x^2+x-6}dx$$
6. $$\int \frac{x^3-11x-26}{x^2-2x-8}dx$$
7. $$\int \frac{1}{x(x^2+1)^2}$$