INTEGRAL CALCULUS (MATH 106)

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1. Half-Angle Substitution

2. Miscelleneous substitutions

3. Improper Integrals

Weekly Objectives

Week 9: Half-Angle Substitution and Improper integrals.

The student is expected to be able to:

1. solve integrals of rational functions involving $\sin x$ or $\cos x$.
2. solve integrals involving fraction powers of x.
3. solve integrals involving a square root of a linear factor.
4. deal with improper integrals.

Half-Angle Substitution

It is used to solve integrals of rational functions involving $\sin x$ or $\cos x$

Example 2.1

$$\int \frac{1}{2+\cos x}dx, \text{ and } \int \frac{1}{1-\sin x}dx$$

Question

How to solve an integral using half angle trigonometric substitution?

Half-Angle Substitution

To solve this type of integral we have to concentrate on:

1. $u = \tan \dfrac{x}{2}$
2. $\sin x = 2\cos\dfrac{x}{2}\sin\dfrac{x}{2} = \dfrac{2\tan\frac{x}{2}}{\sec^2\frac{x}{2}} = \dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \dfrac{2u}{1+u^2}$
3. $\cos x = 2\cos^2\dfrac{x}{2} - 1 = \dfrac{2}{1+\tan^2\frac{x}{2}} - 1 = \dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \dfrac{1-u^2}{1+u^2}$
4. $\dfrac{du}{dx} = \dfrac{1}{2}\sec^2\dfrac{x}{2} = \dfrac{1}{2}(\tan^2\dfrac{x}{2}+1) = \dfrac{1}{2}(u^2+1) \Rightarrow$

$$dx = \frac{2}{(u^2+1)}du$$

Half-Angle Substitution

Example 2.2

Evaluate $\displaystyle\int \frac{1}{2+\cos x}dx$ *and* $\displaystyle\int \frac{1}{1-\sin x}dx$

to solve $\displaystyle\int \frac{1}{2+\cos x}dx$

we put $u = \tan\dfrac{x}{2}$ So $\cos x = \dfrac{1-u^2}{1+u^2}$, and $dx = \dfrac{2}{(u^2+1)}du$

$$\int \frac{1}{2+\cos x}dx = \int \frac{2}{(2+\frac{1-u^2}{1+u^2})(u^2+1)}du = \int \frac{2}{u^2+3}du$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\frac{u}{\sqrt{3}} + c = \frac{2}{\sqrt{3}}\tan^{-1}\frac{\frac{x}{2}}{\sqrt{3}} + c$$

To solve $\displaystyle\int \frac{1}{1-\sin x}dx$

We put $u = \tan\frac{x}{2}, \sin x = \dfrac{2u}{1+u^2}$ and $dx = \dfrac{2}{(u^2+1)}du$

$$\int \frac{1}{1-\sin x}dx = \int \frac{2}{1-\frac{2u}{1+u^2}(1+u^2)}du$$

$$= -\frac{2}{(u-1)} + c = \frac{-2}{tan\frac{x}{2}-1} + c$$

Half-Angle Substitution

Example 2.3

How we can integrate using tangent half angle substitution.

$$\int \frac{1}{3-5\sin x}dx$$

We put $u = \tan\frac{x}{2}$, $\sin x = \frac{2u}{u^2+1}$ and $dx = \frac{2}{u^2+1}$

Hence, the given integral becomes:

$$\int \frac{1}{3-5\sin x}dx = \int \frac{\frac{2}{u^2+1}}{3-5(\frac{2u}{u^2+1})}du = \int \frac{2}{3u^2-10u+3}du$$

Now, we need to do partial fraction decomposition.

Half-Angle Substitution

$$\frac{2}{3u^2 - 10u + 3} = \frac{2}{(u-3)(3u-1)} = \frac{A}{u-3} + \frac{B}{3u-1}$$

$$2 = A(3u-1) + B(u-3) \Rightarrow 2 = (3A+B)u - A - 3B$$

$3A + B = 0$
$-A - 3B = 2$
$A = \frac{1}{4}$, and $B = -\frac{3}{4}$

Half-Angle Substitution

$$\int \frac{1}{3-5\sin x}dx = \int \frac{2}{3u^2-10u+3}du$$

$$= \int \frac{1}{4(u-3)}\,du - \int \frac{3}{4(3u-1)}\,du$$

$$= \frac{1}{4}\ln|u-3| - \frac{3}{4}\frac{1}{3}\ln|3u-1| + c$$

$$\int \frac{1}{3-5\sin x}dx = \frac{1}{4}\ln|\tan^{-1}\frac{x}{2}-3| - \frac{1}{4}\ln|3\tan^{-1}\frac{x}{2}-1| + c$$

- Integrals involving fraction powers of x.

Example 3.1

$\int \frac{1}{x^{\frac{1}{2}} + x^{\frac{1}{3}}} dx$, *to solve this integral we put:*
$u = x^{\frac{1}{6}} \Rightarrow x = u^6 \Rightarrow dx = 6u^5 du$
Note that 6 is the least common multiple of 2 and 3.

$$\int \frac{1}{x^{\frac{1}{2}} + x^{\frac{1}{3}}} dx = 2x^{\frac{1}{2}} - 3x^{\frac{1}{3}} + 6x^{\frac{1}{6}} - 6 \ln \left| x^{\frac{1}{6}} + 1 \right| + c$$

- Integrals involving a square root of a linear factor.

Example 3.2

$\int \frac{1}{(x+1)\sqrt{x-2}}dx$, *to solve this integral we put:*
$u = \sqrt{x-2} \Rightarrow x = u^2 + 2 \Rightarrow dx = 2u\ du$

$$\int \frac{1}{(x+1)\sqrt{x-2}}dx = \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{\sqrt{x-2}}{\sqrt{3}}\right)$$

Improper Integrals with a discontinuous integrand

Definition 4.1

1. *If f is continuous on $[a, b)$ and $|f(x)| \to \infty$ as $x \to b^-$ then*

$$\int_a^b f(x)\ dx = \lim_{t \to b^-} \int_a^t f(x)\ dx$$

2. *If f is continuous on $(a, b]$ and $|f(x)| \to \infty$ as $x \to a^+$ then*

$$\int_a^b f(x)\ dx = \lim_{t \to a^+} \int_t^b f(x)\ dx$$

If the limit exists (and equals a value L) then the improper integral **converges** *(to L).*
If the limit does not exist then the improper integral **diverges**.

Remark

If f is continuous on $[a, b]$ except at a point $c \in (a, b)$ and $|f(x)| \to \infty$ as $x \to c$ then

$$\int_a^b f(x)\,dx = \lim_{t \to c^-} \int_a^t f(x)\,dx + \lim_{t \to c^+} \int_t^b f(x)\,dx$$

If both limits exist (and equals L_1 and L_2 respectively) then the improper integral **converges** (to $L_1 + L_2$).
If at least one of the limits does not exist then the improper integral **diverges**.

Improper Integrals with an infinite limit of integration

Definition 4.2

1. *If f is continuous on* $[a, \infty)$ *then*

$$\int_a^\infty f(x)\, dx = \lim_{t \to \infty} \int_a^t f(x)\, dx$$

2. *If f is continuous on* $(-\infty, a$ *then*

$$\int_{-\infty}^a f(x)\, dx = \lim_{t \to -\infty} \int_t^a f(x)\, dx$$

if the limit exists (and equals a value L) then the improper integral **converges** *(to L).*
If the limit does not exist then the improper integral **diverges**.

Remark

If f is continuous on $(-\infty, \infty)$ then for any constant a

$$\int_{-\infty}^{\infty} f(x)\ dx = \lim_{t\to-\infty} \int_{t}^{a} f(x)\ dx + \lim_{t\to\infty} \int_{a}^{t} f(x)\ dx$$

If both limits exist (and equals L_1 and L_2 respectively) then the improper integral **converges** (to $L_1 + L_2$).
If at least one of the limits does not exist then the improper integral **diverges**.

Examples

1. $\int_1^\infty x^{-2}\,dx$ is an improper integral.

   Some such integrals can sometimes be computed by replacing infinite limits with finite values

   $$\int_1^\infty x^{-2}\,dx = \lim_{y\to\infty}\int_1^y x^{-2}\,dx = \lim_{y\to\infty}\left[-\frac{1}{x}\right]_1^y = \lim_{y\to\infty}\left(-\frac{1}{y}+1\right)$$
   $$= 1$$

2. $$\int_1^\infty \frac{1}{x}\,dx = \lim_{c\to\infty}\int_1^c \frac{1}{x}\,dx = \lim_{c\to\infty}\Big[\ln|x|\Big]_1^c$$
   $$= \lim_{c\to\infty}\Big(\ln|c| - \ln|1|\Big) = \infty$$

Examples

Example 4.1

Evaluate : $\int_{-\infty}^{\infty} \frac{1}{1+X^2}\, dx$

First step:

Split the integral in two.

$$\int_{-\infty}^{\infty} \frac{1}{1+X^2}\, dx = \int_{-\infty}^{0} \frac{1}{1+X^2}\, dx + \int_{0}^{\infty} \frac{1}{1+X^2}\, dx$$

Examples

Second step:
Turn each part into a limit.

$$\int_{-\infty}^{\infty} \frac{1}{1+X^2}\,dx = \lim_{c\to-\infty}\int_{c}^{0} \frac{1}{1+X^2}\,dx + \lim_{c\to\infty}\int_{0}^{c} \frac{1}{1+X^2}\,dx$$

Examples

Third step:

Evaluate each part and add up the results.

$$\int_{-\infty}^{\infty} \frac{1}{1+X^2}\,dx = \lim_{c\to-\infty}\Big[\tan^{-1}x\Big]_c^0 + \lim_{c\to\infty}\Big[\tan^{-1}x\Big]_0^c$$

$$= \lim_{c\to-\infty}\Big[\tan^{-1}0 - \tan^{-1}c\Big] + \lim_{c\to\infty}\Big[\tan^{-1}c - \tan^{-1}0\Big]$$

$$= \left(0-\left(-\tfrac{\pi}{2}\right)\right) + \left(\tfrac{\pi}{2}-0\right) = \pi$$

Examples

Example 4.2

Evaluate $\int_1^\infty \frac{1}{x^p}\, dx$, *where p is a real number.*

We have to consider every possible value of p.

First, for $p = 1$

$$\int_1^\infty \frac{1}{x}\, dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\, dx = \lim_{t\to\infty} \Big[\ln x\Big]_1^t = \lim_{b\to\infty} \ln |t| = \infty$$

so the integral diverges when p = 1.

Examples

Now, for $p \neq 1$, the power rule applies:

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^p}\,dx = \lim_{t\to\infty}\left[\frac{1}{1-p}x^{1-p}\right]_1^t$$

$$= \lim_{t\to\infty}\left(\frac{1}{1-p}t^{1-p} - \frac{1}{1-p}\right) = \frac{1}{1-p}$$

This means that for $p \leq 1$, the integral diverges, and for $p > 1$, it converges and equals $\dfrac{-1}{1-p}$

Examples

Example 4.3

Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+16} dx$

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+16} dx = \lim_{a\to-\infty} \int_{a}^{0} \frac{dx}{x^2+16} dx + \lim_{b\to\infty} \int_{0}^{b} \frac{dx}{x^2+16} dx$$

$$= \lim_{a\to-\infty} \left(0 - \frac{1}{4} tan^{-1} \frac{a}{4}\right) + \lim_{b\to\infty} \left(\frac{1}{4} tan^{-1} \frac{b}{4} - 0\right)$$

$$= \frac{1}{4}\frac{\pi}{2} + \frac{1}{4}\frac{\pi}{2} = \frac{\pi}{4}$$