Discrete Mathematics (MATH 151)

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1. Relations
   - Introduction
   - Relations and Their Properties
   - Representing Relations Using Matrices
   - Equivalence Relations

Relations and Their Properties

The most direct way to express a relationship between elements of two sets is to use ordered pairs made up of two related elements. For this reason, sets of ordered pairs are called binary relations. In this section we introduce the basic terminology used to describe binary relations.

Definition 2.1

Let A and B be a two sets.
A binary relation from A to B is a subset of $A \times B$.

Relations and Their Properties

a binary relation from A to B is a set R of ordered pairs where the first element of each ordered pair comes from A and the second element comes from B.

Remark 2.1

We use the notation $a\ R\ b$ *to denote that* $(a, b) \in R$.

Example 2.1

Let $A = \{0, 1, 2\}$ *and* $B = \{a, b\}$. *Then* $\{(0, a), (0, b), (1, a), (2, b)\}$ *is a relation from* A *to* B.

Relations on a Set

Definition 2.2

A relation on a set A is a relation from A to A.
In other words, a relation on a set A is a subset of $A \times A$.

Example 2.2

Let A be the set $\{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{(a, b)|\ a \text{ divides } b\}$?

Solution

Because (a, b) is in R if and only if a and b are positive integers not exceeding 4 such that a divides b, we see that

$$R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}.$$

Relations on a Set

Example 2.3

Consider these relations on the set of integers:

$R_1 = \{(a, b) | a \leq b\}$,
$R_2 = \{(a, b) | a > b\}$,
$R_3 = \{(a, b) | a = b \text{ or } a = -b\}$,
$R_4 = \{(a, b) | a = b\}$,
$R_5 = \{(a, b) | a = b + 1\}$,
$R_6 = \{(a, b) | a + b \leq 3\}$.

Which of these relations contain each of the pairs (1, 1), (1, 2), (2, 1), (1,-1), and (2, 2)?

Solution

The pair (1, 1) is in R_1, R_3, R_4, and R_6; $(1, 2)$ is in R_1 and R_6; $(2, 1)$ is in R_2, R_5, and R_6; $(1, ?1)$ is in R_2, R_3, and R_6; and finally, (2, 2) is in R, R, and R

Properties of Relations

Definition 2.3

A relation R on a set A is called reflexive if $(a, a) \in R$ for every element $a \in A$.

Definition 2.4

A relation R on a set A is called symmetric if $(b, a) \in R$ whenever $(a, b) \in R$, for all $a, b \in A$.
A relation R on a set A such that for all $a, b \in A$, if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$ is called antisymmetric.

Properties of Relations

Example 2.4

Consider the following relations on $\{1, 2, 3, 4\}$*:*
$R_1 = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)\}$,
$R_2 = \{(1,1),(1,2),(2,1)\}$,
$R_3 = \{(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)\}$,
$R_4 = \{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)\}$,
$R_5 = \{(1,1),(1,2),(1,3),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\}$,
$R_6 = \{(3,4)\}$.
Which of these relations are reflexive, symmetric ?

Solution

The relations R3 and R5 are reflexive
The relations R2 and R3 are symmetric
R4, R5, and R6 are all antisymmetric.

Definition 2.5

A relation R on a set A is called transitive if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, for all $a, b, c \in A$.

Combining Relations

Because relations from A to B are subsets of $A \times B$, two relations from A to B can be combined in any way two sets can be combined.

Example 2.5

Let $A = \{1, 2, 3\}$ *and* $B = \{1, 2, 3, 4\}$. *The relations* $R_1 = \{(1, 1), (2, 2), (3, 3)\}$ *and* $R_2 = \{(1, 1), (1, 2), (1, 3), (1, 4)\}$ *can be combined to obtain*

$R_1 \cup R_2 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (3, 3)\}$,
$R_1 \cap R_2 = |\{(1, 1)\}$,
$R_1 - R_2 = \{(2, 2), (3, 3)\}$,
$R_2 - R_1 = \{(1, 2), (1, 3), (1, 4)\}$.

Definition 2.6

Let R be a relation from a set A to a set B and S a relation from B to a set C. The composite of R and S is the relation consisting of ordered pairs (a, c), where $a \in A, c \in C$, and for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of R and S by $S \circ R$.

Example 2.6

What is the composite of the relations R and S, whereR is the relation from $\{1, 2, 3\}$ to $\{1, 2, 3, 4\}$ with
$R = \{(1, 1), (1, 4), (2, 3), (3, 1), (3, 4)\}$
and S is the relation from $\{1, 2, 3, 4\}$ to $\{0, 1, 2\}$ with
$S = \{(1, 0), (2, 0), (3, 1), (3, 2), (4, 1)\}$?

Solution

$S \circ R = \{(1, 0), (1, 1), (2, 1), (2, 2), (3, 0), (3, 1)\}$.

Definition 2.7

Let R be a relation on the set A. The powers $R^n, n = 1, 2, 3, ...,$ are defined recursively by $R^1 = R$ and $R^{n+1} = R^n \circ R$.

Example 2.7

Let $R = \{(1,1),(2,1),(3,2),(4,3)\}$.
Find the powers $R^n, n = 2, 3, 4, ...$

Solution

Because $R^2 = R \circ R$, we find that
$R^2 = \{(1,1),(2,1),(3,1),(4,2)\}$. Furthermore, because
$R^3 = R^2 \circ R$, $R^3 = \{(1,1),(2,1),(3,1),(4,1)\}$. Additional computation shows that R^4 is the same as R^3,
so $R^4 = \{(1,1),(2,1),(3,1),(4,1)\}$.

Theorem 2.1

The relation R on a set A is transitive if and only if $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$.

Representing Relations Using Matrices

Representing Relations using Zero-One Matrices:
Let R be a relation from
$A = \{a_1, a_2, ..., a_m\}$ to $B = \{b_1, b_2, ..., b_n\}$.
The relation R can be represented by the matrix $M_R = [m_ij]$, where

$$M_R = \begin{cases} 1 & if\ (a_i, b_j) \in R \\ 0 & if\ (a_i, b_j) \notin R \end{cases}$$

Representing Relations Using Matrices

Example 2.8

Suppose that $A = \{1, 2, 3\}$ *and* $B = \{1, 2\}$.
Let R *be the relation from* A *to* B *containing* (a, b) ,
Where $a \in A, b \in B$,

$$R = \{(2, 1), (3, 1), (3, 2)\}$$

What is the matrix representing R *?*

Solution

$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Example 2.9

Let $A = \{a_1, a_2, a_3\}$ *and* $B = \{b_1, b_2, b_3, b_4, b_5\}$. *Which ordered pairs are in the relation* R *represented by the matrix :*

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Solution

$R = \{(a_1, b_2), (a_2, b_1), (a_2, b_3), (a_3, b_1), (a_3, b_3), (a_3, b_4)\}$

Reflexive in a Zero-One Matrix

Let R be a binary relation on a set and let M_R be its zero-one matrix. R is reflexive if and only if $M_{i,i} = 1$ for all i. In other words, all elements are equal to 1 on the main diagonal.

$$M_R = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}$$

Symmetric in a Zero-One Matrix

Let R be a binary relation on a set and let M_R be its zero-one matrix. R is symmetric if and only if $M_R = M_R^t$.
In other words, $M_{i,j} = M_{j,i}$ for all i and j.

$$M_R = \begin{bmatrix} \ddots & 1 & & \\ 1 & \ddots & 0 & \\ & 0 & \ddots & 1 \\ & & 1 & \ddots \end{bmatrix}$$

Antisymmetric in a Zero-One Matrix

Let R be a binary relation on a set and let M_R be its zero-one matrix.

R is antisymmetric if and only if $M_{i,j} = 0$ or $M_{j,i} = 0$ for all $i \neq j$.

$$M_R = \begin{bmatrix} \ddots & 1 & & \\ 0 & \ddots & 0 & \\ & 0 & \ddots & 0 \\ & & 1 & \ddots \end{bmatrix}$$

Example

Example 2.10

Suppose that the relation R on a set is represented by the matrix

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Is R reflexive, symmetric, and/or antisymmetric?

Solution

Because all the diagonal elements of this matrix are equal to 1, R is reflexive. Moreover, because M_R is symmetric, it follows that R is symmetric. It is also easy to see that R is not antisymmetric.

Suppose that R_1 and R_2 are relations on a set A represented by the matrices M_{R_1} and M_{R_2} , respectively.
The matrix representing the union of these relations has a 1 in the positions where either M_{R_1} or M_{R_2} has a 1.
The matrix representing the intersection of these relations has a 1 in the positions where both M_{R_1} and M_{R_2} have a 1.
Thus, the matrices representing the union and intersection of these relations are

$$M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2} \text{ and } M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2}$$

Example

Example 2.11

Suppose that the relations R_1 and R_2 on a set A are represented by the matrices:

$$M_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

What are the matrices representing $R_1 \cup R_2$ and $R_1 \cap R_2$?

Solution

The matrices of these relations are

$$M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The matrix representing the composite of two relations can be used to find the matrix for M_{R^n}. In particular, $M_{R^n} = {M^n}_R$

Example 2.12

Find the matrix representing the relation R^2, where the matrix representing R is $M_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

Solution

$$M_{R^2} = M_R^2 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

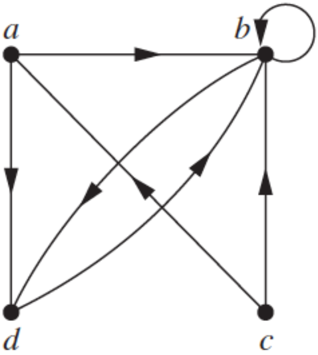
Representing Relations Using Digraphs

Definition 2.8

A directed graph, or digraph, consists of a set V of vertices (or nodes) together with a set E of ordered pairs of elements of V called edges (or arcs). The vertex a is called the initial vertex of the edge (a, b), and the vertex b is called the terminal vertex of this edge.

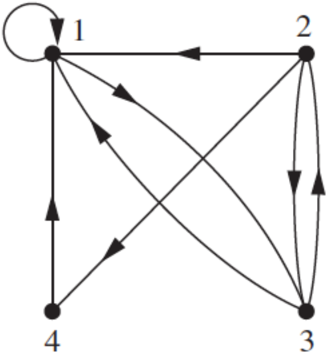
Example

The directed graph with vertices a, b, c, and d, and edges $(a, b), (a, d), (b, b), (b, d), (c, a), (c, b)$, and (d, b) is displayed in Figure



Example

The directed graph of the relation $R = \{(1,1),(1,3),(2,1),(2,3),(2,4),(3,1),(3,2),(4,1)\}$ on the set $\{1,2,3,4\}$ is shown in Figure



Definition 2.9

A relation on a set A is called an equivalence relation if it is reflexive, symmetric, and transitive.

Definition 2.10

Two elements a and b that are related by an equivalence relation are called equivalent .
The notation $a \sim b$ is often used to denote that a and b are equivalent elements with respect to a particular equivalence relation.

Example

Let $A = \mathbb{Z}$ and define $R = \{(x, y)|x \text{ and } y \text{ have the same parity}\}$ i.e., x and y are either both even or both odd. The parity relation is an equivalence relation.

1. For any $x \in \mathbb{Z}$, x has the same parity as itself, so $(x, x) \in R$.
2. If $(x, y) \in R, x$ and y have the same parity, so $(y, x) \in R$.
3. If $(x, y) \in R$, and $(y, z) \in R$, then x and z have the same parity as y, so they have the same parity as each other (if y is odd, both x and z are odd; if y is even, both x and z are even), thus $(x, z) \in R$.

Example

Let $A = \mathbb{R}$ and define the "square" relation $R = \{(x, y)|x^2 = y^2\}$.
The square relation is an equivalence relation.

1. For all $x \in \mathbb{R}, x^2 = x^2,\ so\ (x, x) \in R$.
2. If $(x, y) \in R, x^2 = y^2, so\ y^2 = x^2\ and\ (y, x) \in R$.
3. If $(x, y) \in R$ and $(y, z) \in R$ then $x^2 = y^2 = z^2$, so $(x, z) \in R$.

Equivalence Classes

Definition 2.11

Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the equivalence class of a. The equivalence class of a with respect to R is denoted by $[a]_R$. When only one relation is under consideration, we can delete the subscript R and write $[a]$ for this equivalence class.

$$[a] = \{s|(a,s) \in R\} = \{s|aRs\}$$

Example

Example 2.13

Let $A = \{1, 2, 3\}$ *and* $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$.
Then $[1] = \{1, 2\}$, $[2] = \{1, 2\}$ *and* $[3] = \{3\}$.

Example 2.14

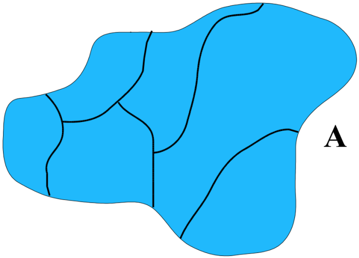
Let $A = \mathbb{R}$ *and* $R = \{(x, y) | x^2 = y^2\}$.
Then $[0] = \{0\}$, $[1] = \{1, -1\}$, $[\frac{1}{4}] = \{\frac{1}{4}, \frac{-1}{4}\}$, *and* $[x] = \{x, -x\}$.

Equivalence Classes and Partitions

Partitions

A partition of a set A is a family F of non-empty subsets of A such that:

1. If B_1 and B_2 then either $B_1 = B_2$ or $B_1 \cap B_2 = \phi$.
2. $\cup_{B_i \in F} = A$



Equivalence Classes and Partitions

Example 2.15

Suppose that $A = \{1, 2, 3, 4, 5, 6\}$. *The collection of sets* $B_1 = \{1, 2, 3\}, B_2 = \{4, 5\}$, *and* $B_3 = \{6\}$ *forms a partition of* S, *because these sets are disjoint and their union is* A.

Theorem 2.2

Let R *be an equivalence relation on a set* A. *These statements for elements* a *and* b *of* A *are equivalent:*
(i) $a \; R \; b$
(ii) $[a] = [b]$
(iii) $[a] \cap [b] = \phi$

Equivalence Classes and Partitions

Theorem 2.3

Let R be an equivalence relation on a set A. Then the equivalence classes of R form a partition of A. Conversely, given a partition $\{B_i | i \in I\}$ of the set A, there is an equivalence relation R that has the sets $B_i, i \in I$, as its equivalence classes.

Partial Orderings

Definition 2.12

A relation R on a set S is called a partial ordering or partial order if it is reflexive, antisymmetric, and transitive. A set S together with a partial ordering R is called a partially ordered set, or poset, and is denoted by (S,R). Members of S are called elements of the poset.

Example 2.16

Show that the greater than or equal relation $(\geq)$ *is a partial ordering on the set of integers.*

elements comparable

Definition 2.13

The elements a and b of a poset $(S, \preceq)$ *are called comparable if either* $a \preceq b$ *or* $b \preceq a$*. When a and b are elements of S such that neither* $a \preceq b$ *nor* $b \preceq a$*, a and b are called incomparable.*

Example 2.17

In the poset $(Z^+, |)$*, are the integers 3 and 9 comparable? Are 5 and 7 comparable?*

totally ordered

Definition 2.14

If $(S, \preceq)$ is a poset and every two elements of S are comparable, S is called a totally ordered or linearly ordered set, and is called a total order or a linear order. A totally ordered set is also called a chain.

Example 2.18

The poset $(Z, \leq)$ is totally ordered, because $a \leq b$ or $b \leq a$ whenever a and b are integers.

well-ordered

Definition 2.15

$(S, \preceq)$ *is a* well-ordered *set if it is a poset such that is a total ordering and every nonempty subset of* S *has a least element.*

Example 2.19

The set of ordered pairs of positive integers, $Z^+ \times Z^+$, *with* $(a_1, a_2) \preceq (b_1, b_2)$ *if* $a_1 < b_1$, *or if* $a_1 = b_1$ *and* $a_2 \leq b_2$ *(the lexicographic ordering), is a well-ordered set.*

Lexicographic Order

we will show how to construct a partial ordering on the Cartesian product of two posets, $(A_1, \preceq_1)$ and $(A_2, \preceq_2)$.
The lexicographic ordering on $A_1 \times A_2$ is defined by specifying that one pair is less than a second pair if the first entry of the first pair is less than (in A_1) the first entry of the second pair, or if the first entries are equal, but the second entry of this pair is less than (in A_2) the second entry of the second pair. In other words, (a_1, a_2) is less than (b_1, b_2), that is,
$(a_1, a_2) \prec (b_1, b_2)$,
either if $a_1 \prec_1 b_1$ or if both $a_1 = b_1$ and $a_2 \prec_2 b_2$.

Lexicographic Order

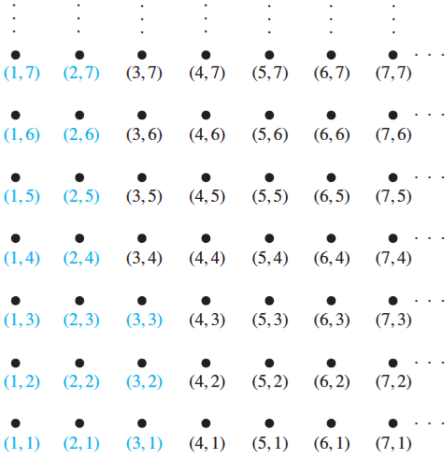
Example 2.20

Determine whether $(3,5) \prec (4,8)$, *whether* $(3,8) \prec (4,5)$, *and whether* $(4,9) \prec (4,11)$ *in the poset* $(\mathbb{Z} \times \mathbb{Z})$, *where is the lexicographic ordering constructed from the usual* $\leq$ *relation on* $\mathbb{Z}$.

Solution

Because $3 < 4$, it follows that $(3,5) \prec (4,8)$ and that $(3,8) \prec (4,5)$. We have $(4,9) \prec (4,11)$, because the first entries of $(4,9)$ and $(4,11)$ are the same but $9 < 11$.

Lexicographic Order



Hasse Diagrams

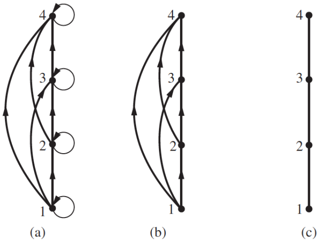
A Hasse diagram is a graphical rendering of a partially ordered set displayed via the cover relation of the partially ordered set with an implied upward orientation. A point is drawn for each element of the poset, and line segments are drawn between these points according to the following two rules:

1. If $x < y$ in the poset, then the point corresponding to x appears lower in the drawing than the point corresponding to y.
2. The line segment between the points corresponding to any two elements x and y of the poset is included in the drawing iff x covers y or y covers x.

Hasse Diagrams

Example 2.21

Consider the directed graph for the partial ordering $\{(a, b) | a \leq b\}$ *on the set* $\{1, 2, 3, 4\}$



Hasse Diagram