King Saud University: Math. Dept. Math-254
Semester II 1437-38 H Final Exam. Time: 3 Hours

Question 1: (4 + 5 + 5)

(a) Find a bound for the number of iterations needed to achieve an approximation with accuracy 10^{-1} to the solution of $xe^x = 1$ lying in the interval $[0.5, 1]$ using the bisection method. Find the approximation to the root with this degree of accuracy.

(b) Show that Newton's iterative method for computing $(a)^{1/k}$ is given by

$$x_{n+1} = \frac{1}{k}\left[(k-1)x_n + \frac{a}{x_n^{k-1}}\right], \quad n \geq 0.$$

Use this iterative scheme to find second approximation of $(32)^{1/5}$, $x_0 = 1.5$. Compute absolute error.

(c) Let $f(x) = e^{(x+2)}$ and $x_0 = -0.2, x_1 = -0.1, x_2 = 0, x_3 = 0.1, x_4 = 0.2, x_5 = 0.3$. If $p_4(x) = 7.7679002$ and $f[-0.2, -0.1, 0, 0.1, 0.2, 0.3] = 0.0649$, then find the approximation of $e^{(2.05)}$ using fifth degree Newton's interpolating polynomial. Compute error bound and absolute error.

Question 2: (5 + 4 + 4)

(a) Let $f(x) = x^3 + 1$ defined on the interval $[0.1, 0.2]$. Find the value of unknown point $\eta(x)$ such that the error term for the simple Trapezoidal rule is equal to the exact error.

(b) Use the best composite integration formula to approximate the integral $\int_0^1 \frac{dx}{7-2x}$, with $h = 0.25$. Estimate the error bound.

(c) The function $f(x)$ satisfies a given equation $f''(x) = x^2 f(x)$ and satisfy the conditions $f(0.5) = 2,\ f(0.7) = 4$. Use the central-difference formula to find approximation of $f''(0.6)$.

Question 3: (4 + 5 + 4)

Consider the following linear system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 4 & -1 & 0 \\ -1 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

(a) Use simple Gauss elimination to find A^{-1} and then use it to find a unique solution.

(b) Find the number of iterations k needed to get an accuracy within 10^{-4} for solving the given system using Gauss-Seidel iterative method when $\mathbf{x}^{(0)} = [0.5, 0.3, 0.2]^T$.

(c) If $f(x) = \frac{1}{x}$, then show that $f[1, 1, 1, 2] = -\frac{1}{2}$.

Solution Q1(a). Here $a = 0.5$, $b = 1$ and $k = 1$, then

$$n \geq \frac{\ln[10^1(1-0.5)]}{\ln 2} \approx 2.3219, \quad n = 3.$$

The given function $f(x) = xe^x - 1$ is continuous on $[0.5, 1.0]$, so starting with $a_1 = 0.5$ and $b_1 = 1$, we compute:

$$\begin{array}{rcll} a_1 & = & 0.5: & f(a_1) = -0.1756, \\ b_1 & = & 1: & f(b_1) = 1.7183, \end{array}$$

since $f(0.5)f(1) < 0$, so that a root of $f(x) = 0$ lies in the interval $[0.5, 1]$. Then

$$c_1 = \frac{a_1 + b_1}{2} = 0.75; \qquad f(c_1) = 0.5878.$$

Hence the function changes sign on $[a_1, c_1] = [0.5, 0.75]$. To continue, we squeeze from right and set $a_2 = a_1$ and $b_2 = c_1$. Then

$$c_2 = \frac{a_2 + b_2}{2} = 0.625; \qquad f(c_2) = 0.1677.$$

Finally, we have in the similar manner as

$$c_3 = \frac{a_3 + b_3}{2} = 0.5625.$$

Solution Q1(b). Given $x = a^{1/k}$, so

$$x^k - a = 0,$$

Let

$$f(x) = x^k - a \quad \text{and} \quad f'(x) = kx^{k-1}.$$

Hence, assuming an initial estimate to the root, say, $x = x_0$, we get

$$x_1 = x_0 - \frac{(x_0^k - a)}{kx_0^{k-1}} = x_0 - \frac{x_0^k}{kx_0^{k-1}} + \frac{a}{kx_0^{k-1}} = \frac{1}{k}\left[(k-1)x_0 + \frac{a}{x_0^{k-1}}\right], \quad n \geq 0.$$

In general, we have

$$x_{n+1} = \frac{1}{k}\left[(k-1)x_n + \frac{a}{x_n^{k-1}}\right], \quad n \geq 0.$$

Since we want the approximations of the fifth root of number 32, so we take $a = 32$ and $k = 5$. Given the initial approximation $x_0 = 1.5$, then by using this iterative formula, we get

$$x_1 = 2.4642 \qquad \text{and} \qquad x_2 = 2.1449,$$

and absolute error is

$$Abs - Error = |2 - 2.1449| = 0.1449.$$

Solution Q1(c). Since the fifth-degree Newton polynomial $p_5(x)$ is defined as

$$f(x) = p_5(x) = p_4(x) + (x - x_0)(x - x_1)(x - x_2)(x - x_3)(x - x_4)f[x_0, x_1, x_2, x_3, x_4, x_5],$$

and using the given data points, we have

$$e^{2.05} \approx p_5(0.05) = 7.7679002 + (0.05+0.2)(0.05+0.1)(0.05-0.0)(0.05-0.1)(0.05-0.2)(0.0649) = 7.7679011.$$

To compute an error bound for the approximation of the given function in the interval $[-0.2, 0.3]$, we use the following error formula

$$|f(x) - p_5(x)| \leq \frac{M}{6!}|(0.05+0.2)(0.05+0.1)(0.05-0.0)(0.05-0.1)(0.05-0.2)(0.05-0.3)|.$$

Since

$$|f^{(6)}(\eta(x))| \leq M = \max_{-0.2 \leq x \leq 0.3} |f^{(6)}(x)| = \max_{-0.2 \leq x \leq 0.3} |e^{x+2}| = 9.9742,$$

so

$$|f(0.05) - p_5(0.05)| \leq \frac{9.9742}{720}(3.5156 \times 10^{-6}) = 4.8702 \times 10^{-8}.$$

which is desired error bound. Also, we have to compute absolute error as

$$|f(0.05) - p_5(0.05)| = |e^{2.05} - p_5(0.05)| = |7.7679011 - 7.7679011| = 0.$$

Solution Q2(a). Given $f(x) = x^3 + 1$, and $[a, b] = [0.1, 0.2]$, we use the formula of the Trapezoidal rule for $h = 0.1$, as follows

$$ApproxValue = \frac{0.1}{2}\Big[f(0.1) + f(0.2)\Big] = \frac{0.1}{2}\Big[[(0.1)^3 + 1] + [(0.2)^3 + 1]\Big] = 0.10045.$$

We know that

$$ExactValue = \int_{0.1}^{0.2} (x^3 + 1)\, dx = (x^4/4 + x)\Big|_{0.1}^{0.2} = [(0.2)^4/4 + 0.2] - [(0.1)^4/4 + 0.1] = 0.100375,$$

so we have the error

$$E = (ExactValue) - (ApproxValue) = 0.100375 - 0.10045 = -0.000075.$$

since the second derivative of the given function is $f''(x) = 6x$, so by using the local error for the T Since the fourth derivative of the function is

$$f^{(4)}(x) = \frac{384}{(7-2x)^5}.$$

and

$$M = \max_{0 \leq x \leq 1} |f^{(4)}(x)| = 0.1229.$$

Thus the error bound is

$$|E_{S_4}(f)| \leq \frac{(0.1229)(0.25)^4}{180} = 2.667 \times 10^{-6}.$$

Solution Q2(c). Let $x_0 = (x_1 - h) = 0.5, x_1 = 0.6$, and $x_2 = (x_1 + h) = 0.7$, gives $h = 0.1$, so

$$f''(0.6) = (0.6)^2 f(0.6) \approx \frac{f(0.5) - 2f(0.6) + f(0.7)}{0.01},$$

Using error term of Trapezoidal rule, we have

$$-0.000075 = -\frac{(0.1)^3}{12}(6\eta(x)),$$

gives the value of $\eta(x) = 0.15$.

Solution Q2(b). Since $h = 0.25$, so $n = \dfrac{1-0}{0.25} = 4$. By using the Simpson's composite formula, we have

$$\int_0^1 f(x)\,dx \approx \frac{0.25}{3}\Big[f(0) + 4[f(0.25) + f(0.75)] + 2f(0.5) + f(1)\Big]$$

Thus

$$\int_0^1 f(x)\,dx \approx \frac{0.25}{3}\Big[0.1429 + 4[0.1539 + 0.1818] + 2(0.16667) + 0.2\Big] \approx 0.1682.$$

which is equal to

$$\begin{aligned} (0.01)(0.6)^2 f(0.6) &\approx [2 - 2f(0.6) + 4] \\ f(0.6) &\approx 2.9946. \end{aligned}$$

Thus

$$f''(0.6) \approx \frac{[2 - 2(2.9946) + 4]}{0.01} \approx 1.0781.$$

Solution Q3(a). Suppose that the inverse $A^{-1} = B$ of the given matrix exists and let

$$AB = \begin{pmatrix} 4 & -1 & 0 \\ -1 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I}.$$

Now to find the elements of the matrix B, we apply the simple Gaussian elimination on the augmented matrix

$$[A|\mathbf{I}] = \left(\begin{array}{rrr|rrr} 4 & -1 & 0 & 1 & 0 & 0 \\ -1 & 3 & 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array}\right).$$

$$\left(\begin{array}{rrr|rrr} 4 & -1 & 0 & 1 & 0 & 0 \\ 0 & 11/4 & 1 & 1/4 & 1 & 0 \\ 0 & 0 & 29/11 & -1/11 & -4/11 & 1 \end{array}\right).$$

We solve the first system

$$\begin{pmatrix} 4 & -1 & 0 \\ 0 & 11/4 & 1 \\ 0 & 0 & 29/11 \end{pmatrix} \begin{pmatrix} b_{11} \\ b_{21} \\ b_{31} \end{pmatrix} = \begin{pmatrix} 1 \\ 1/4 \\ -1/11 \end{pmatrix},$$

which gives $b_{11} = 8/29,\ b_{21} = 3/29,\ b_{31} = -1/29$. Similarly, the solution of the second linear system

$$\begin{pmatrix} 4 & -1 & 0 \\ 0 & 11/4 & 1 \\ 0 & 0 & 29/11 \end{pmatrix} \begin{pmatrix} b_{12} \\ b_{22} \\ b_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -4/11 \end{pmatrix},$$

which gives $b_{12} = 3/29,\ b_{22} = 12/29,\ b_{32} = -4/29$. Finally, the solution of the third linear system

$$\begin{pmatrix} 4 & -1 & 0 \\ 0 & 11/4 & 1 \\ 0 & 0 & 29/11 \end{pmatrix} \begin{pmatrix} b_{13} \\ b_{23} \\ b_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

and it gives $b_{13} = -1/29,\ b_{23} = -4/29,\ b_{33} = 11/29$. Hence the elements of the inverse matrix B are

$$B = A^{-1} = \frac{1}{29}\begin{pmatrix} 8 & 3 & -1 \\ 3 & 12 & -4 \\ -1 & -4 & 11 \end{pmatrix},$$

which is the required inverse of the given matrix A.
Thus

$$\mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{29}\begin{pmatrix} 8 & 3 & -1 \\ 3 & 12 & -4 \\ -1 & -4 & 11 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 13/29 \\ 23/29 \\ 2/29 \end{pmatrix} = \begin{pmatrix} 0.4483 \\ 0.7931 \\ 0.0690 \end{pmatrix}.$$

Solution Q3(b). Since the Gauss-Seidel iteration matrix is defined as

$$T_G = -(D+L)^{-1}U = \begin{pmatrix} 0 & -\frac{1}{4} & 0 \\ 0 & -\frac{1}{12} & \frac{1}{3} \\ 0 & \frac{1}{36} & -\frac{1}{9} \end{pmatrix}.$$

Then the l_∞ norm of the matrix T_G is

$$\|T_G\|_\infty = \max\left\{\frac{1}{4}, \frac{5}{36}, \frac{5}{12}\right\} = \frac{5}{12} < 1.$$

The Gauss-Seidel method for the given system is

$$\begin{aligned} x_1^{(k+1)} &= \frac{1}{4}\Big[1 + x_2^{(k)}\Big] \\ x_2^{(k+1)} &= \frac{1}{3}\Big[2 + x_1^{(k+1)} - x_3^{(k)}\Big] \\ x_3^{(k+1)} &= \frac{1}{3}\Big[1 - x_2^{(k+1)}\Big] \end{aligned}$$

Starting with initial approximation $x_1^{(0)} = 0.5, x_2^{(0)} = 0.3, x_3^{(0)} = 0.2$, and for $k = 0$, we obtain the first approximation as

$$\mathbf{x^{(1)}} = [0.325, 0.708, 0.097]^T.$$

To find the number of iterations, we do as

$$\|\mathbf{x} - \mathbf{x^{(k)}}\| \le \frac{\|T_G\|^k}{1 - \|T_G\|}\|\mathbf{x}^{(1)} - \mathbf{x^{(0)}}\| \le 10^{-4},$$

it gives

$$\frac{(5/12)^k}{1 - 5/12}(0.408) \le 10^{-4}.$$

Taking ln on both sides and simplify, we obtain

$$k \ge 10.1117, \quad k = 11.$$

Solution Q3(c). Given $f(x) = \dfrac{1}{x}$ and so $f'(x) = -\dfrac{1}{x^2}$, $f''(x) = \dfrac{2}{x^3}$. Thus $f[1,1,1,2]$ gives:

$$\begin{aligned}
f[1,1,1,2] &= \frac{f[1,1,2]-f[1,1,1]}{2-1} = f[1,1,2] - \frac{f''(1)}{2!} \\
&= \frac{f[1,2]-f[1,1]}{2-1} - -\frac{f''(1)}{2} \\
&= \frac{f[2]-f[1]}{2-1} - \frac{f'(1)}{1!} - -\frac{f''(1)}{2} \\
&= \frac{1}{2} - \frac{1}{1} + 1 - \frac{2}{2} \\
&= -\frac{1}{2}.
\end{aligned}$$