King Saud University
College of Sciences
Department of Mathematics

Final Exam, S1 1439/1440
M 380 – Stochastic Processes
Time: 3 hours – Male Section

Answer the following questions:

Q1: [4+4]

(a) Suppose $X \sim$ Bin (p, N) and $N \sim$ Poisson (λ). Find the marginal probability mass function for X and determine the mean for X.

(b) For the Markov process $\{X_t\}$, t=0,1,2,...,n with states $i_0, i_1, i_2, \ldots, i_{n-1}, i_n$

Prove that: $\Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_n = i_n\} = p_{i_0} P_{i_0 i_1} P_{i_1 i_2} \cdots P_{i_{n-1} i_n}$ where $p_{i_0} = \text{pr}\{X_0 = i_0\}$

Q2: [2+4]

a) A Markov chain $\{X_n\}$ on the states 0, 1, 2 has the transition probability matrix

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} 0.1 & 0.1 & 0.8 \\ 0.2 & 0.2 & 0.6 \\ 0.3 & 0.2 & 0.5 \end{array} \right\| \end{array}$$

Find $pr\{X_3 = 1 | X_0 = 0\}$

b) Consider a spare parts inventory model in which either 0, 1, or 2 repair parts are demanded in any period, with $\Pr\{\xi_n = 0\} = 0.5$, $\Pr\{\xi_n = 1\} = 0.4$, $\Pr\{\xi_n = 2\} = 0.1$ and suppose s=0 and S=2. Determine the transition probability matrix for the Markov chain $\{X_n\}$, where X_n is defined to be the quantity on hand at the end of period n.

Q3: [8]

An airline reservation system has two computers, only one of which is in operation at any given time. A computer may break down on any given day with probability p. There is a duplicate repair facility that takes 2 days to restore a computer to normal. The facilities are such that both two computers can be repaired simultaneously. Form a Markov chain by taking as states the pairs (x,y), where x is the number of machines in operating condition at the end of a day and y is 1 if a day's labor has been expended on a machine not yet repaired and 0 otherwise. Also, find the system availability.

Q4: [5+4]

(a) Let $\{X_n\}$, n=1,2,... be a Markov chain with transition probability matrix

$$P = \begin{array}{c|ccc} & O & D & R \\ \hline O & 0.9 & 0.1 & 0 \\ D & 0 & 0.9 & 0.1 \\ R & 1 & 0 & 0 \end{array}$$

Where X_n denote the condition of a machine of nth period with $X_n = 1$ means "operating", $X_n = 2$ means "deterioration" and $X_n = 3$ means "repairing". Find each of the following:

i) $\Pr\{X_4 = 1\}$, knowing that the process starts in state $X_0 = 1$

ii) The limiting distribution

iii) The long run rate of repairs per unit time.

(b) Messages arrive at a telegraph office as a Poisson process with mean rate of 3 messages per hour.

(i) What is the probability that no messages arrive during the morning hours 8:00 A.M. to noon?

(ii) What is the distribution of the time at which the first afternoon message arrives?

Q5: [5+4]

(a) If X(t) represents a size of a population where X(0)=1, using the following differential equations

$$\frac{dp_0(t)}{dt} = -\lambda_0 p_0(t) \qquad (1)$$

$$\frac{dp_n(t)}{dt} = \lambda_{n-1} p_{n-1}(t) - \lambda_n p_n(t), \quad n=1,2,3,\ldots \qquad (2)$$

Prove that:

$X(t) \sim \text{geom}(p)$, $p = e^{-\lambda t}$ when $\lambda_0 = 0$ and $\lambda_n = n\lambda$, and then find the mean and variance of this process.

(b) Let X(t) be a Yule process that is observed at a random time U, where U is uniformly distributed over [0,1). Show that $pr\{X(U) = k\} = p^k / (\beta k)$ for $k = 1,2,\ldots$, with $p = 1 - e^{-\beta}$.

Model answer of Final Exam

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4+4

Q1 a) $X \sim Bin(p, N)$,

$N \sim poisson(\lambda)$

The conditional prob. mass fn is

$$P_{X|N}(x|n) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0,1,2,\dots,n$$ (1)

and the marginal prob. mass fn

$$P_N(n) = \frac{e^{-\lambda}\lambda^n}{n!}, \quad n = 0,1,2,\dots$$

$$pr(X=x) = \sum_{n=0}^{\infty} P_{X|N}(x|n)\, P_N(n)$$ (1)

$$pr(X=x) = \sum \binom{n}{x} p^x (1-p)^{n-x} \frac{e^{-\lambda}\lambda^n}{n!}$$

$$= p^x e^{-\lambda} \sum_{n=x}^{\infty} \frac{\cancel{n!}}{x!(n-x)!} (1-p)^{n-x} \frac{\lambda^n}{\cancel{n!}}$$

$$pr(X=x) = \frac{\lambda^x e^{-\lambda} p^x}{x!} \cdot \sum_{n=x}^{\infty} \frac{\lambda^{n-x}(1-p)^{n-x}}{(n-x)!}$$

$$pr(X=x) = \frac{(\lambda p)^x e^{-\lambda}}{x!} \sum_{n=x}^{\infty} \frac{[\lambda(1-p)]^{n-x}}{(n-x)!}$$

$$pr(X=x) = \frac{(\lambda p)^x e^{-\lambda}}{x!} \cdot \underbrace{\sum_{r=0}^{\infty} \frac{[\lambda(1-p)]^r}{r!}}_{e^{\lambda(1-p)}}$$

$$\therefore pr(X=x) = \frac{(\lambda p)^x e^{-\lambda p}}{x!}, \quad x = 0,1,2,\dots$$

$\Rightarrow X \sim poisson(\lambda p)$ with mean λp (2)

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b) $pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \dots, X_n = i_n\}$

$= pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \dots, X_{n-1} = i_{n-1}\}$

$\cdot\, pr\{X_n = i_n | X_0 = i_0, X_1 = i_1, X_2 = i_2, \dots, X_{n-1} = i_{n-1}\}$ (1) ①

$pr\{X_n = i_n | X_0 = i_0, X_1 = i_1, \dots, X_{n-1} = i_{n-1}\}$

$= pr\{X_n = i_n | X_{n-1} = i_{n-1}\}$ (1) ②

$= P_{i_{n-1} i_n}$ Defn of Markov process

Subs. ② in ①

$pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \dots, X_n = i_n\}$

$= pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \dots, X_{n-1} = i_{n-1}\} \cdot P_{i_{n-1} i_n}$ (1)

By repeating this argument n-1 times

$pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \dots, X_n = i_n\}$

$= P_{i_0} P_{i_0 i_1} P_{i_1 i_2} \cdots P_{i_{n-2} i_{n-1}} P_{i_{n-1} i_n}$ (1)

where $P_{i_0} = pr\{X_0 = i_0\}$ for initial distn.

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2

Q2 6 (2+4)

a) $P = \begin{bmatrix} 0.1 & 0.1 & 0.8 \\ 0.2 & 0.2 & 0.6 \\ 0.3 & 0.2 & 0.5 \end{bmatrix}$

$Pr\{X_3 = 1 \mid X_0 = 0\} = P^3_{01}$ 1

$$P^2 = \begin{bmatrix} 0.1 & 0.1 & 0.8 \\ 0.2 & 0.2 & 0.6 \\ 0.3 & 0.2 & 0.5 \end{bmatrix} \begin{bmatrix} 0.1 & 0.1 & 0.8 \\ 0.2 & 0.2 & 0.6 \\ 0.3 & 0.2 & 0.5 \end{bmatrix}$$

$$= \begin{bmatrix} 0.27 & 0.19 & 0.54 \\ 0.24 & 0.18 & 0.58 \\ 0.22 & 0.17 & 0.61 \end{bmatrix}$$

$$P^3_{01} = [0.27 \quad 0.19 \quad 0.54] \begin{bmatrix} 0.1 \\ 0.2 \\ 0.3 \end{bmatrix}$$

$$\boxed{P^3_{01} = 0.173}$$ 1

b)
$$P = \begin{array}{c} \\ -1 \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cccc} -1 & 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 0 & 0.1 & 0.4 & 0.5 \\ 0 & 0.1 & 0.4 & 0.5 \\ 0.1 & 0.4 & 0.5 & 0 \\ 0 & 0.1 & 0.4 & 0.5 \end{bmatrix} \end{array}$$

4

$P_{-1-1} = Pr\{X_{n+1} = -1 \mid X_n = -1\}$ $\quad \xi_{n+1} = s - j$, $i \le s$ replenishment

$= pr(\xi_{n+1} = 3) = 0$

, $P_{10} = pr(\xi_{n+1} = 2) = 0.1$

, $P_{21} = pr(\xi_{n+1} = 1) = 0.4$ $\quad \xi_{n+1} = i - j$ non replenishment, $s < i \le S$

, $P_{22} = pr(\xi_{n+1} = 0) = 0.5$

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Q3 8

	(2,0)	(1,0)	(1,1)	(0,1)
(2,0)	q	p	0	0
(1,0)	0	0	q	p
(1,1)	q	p	0	0
(0,1)	0	0	1	0
	π_0	π_1	π_2	π_3

4

$\Rightarrow q\pi_0 + q\pi_2 = \pi_0 \Rightarrow \pi_2 = \frac{p}{q}\pi_0$

, $p\pi_0 + p\pi_2 = \pi_1 \Rightarrow \pi_1 = \frac{p}{q}\pi_0$

$\therefore \pi_1 = \pi_2 = \boxed{\frac{p}{q}\pi_0}$ ① 1

, $p\pi_1 = \pi_3 \Rightarrow \pi_3 = \boxed{\frac{p^2}{q}\pi_0}$ ② 1

, $\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1$ ③ 1

$\Rightarrow \boxed{\pi_0 = \frac{q}{1+p+p^2}}$

$\Rightarrow \boxed{\pi_3 = \frac{p^2}{1+p+p^2}}$ ④

the limiting distn is

$\pi = \left(\frac{q}{1+p+p^2}, \frac{p}{1+p+p^2}, \frac{p}{1+p+p^2}, \frac{p^2}{1+p+p^2}\right)$

the availability, the prob that at least one Computer is operating 1

$= 1 - \pi_3$

$= 1 - \frac{p^2}{1+p+p^2} = \boxed{\frac{1+p}{1+p+p^2}}$

(for two repair facility)

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3/

Q4

a)
$$P = \begin{array}{c} \\ O \\ D \\ R \end{array}\begin{array}{c} \begin{array}{ccc} O & D & R \end{array} \\ \begin{bmatrix} 0.9 & 0.1 & 0 \\ 0 & 0.9 & 0.1 \\ 1 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ \pi_1 & \pi_2 & \pi_3 \end{array} \end{array}$$

9

i) $pr\{X_4 = 1\}, X_0 = 1$

$= pr\{X_4 = 1 \mid X_0 = 1\}$

$= P_{11}^4$

$$P^2 = \begin{bmatrix} 0.9 & 0.1 & 0 \\ 0 & 0.9 & 0.1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0.9 & 0.1 & 0 \\ 0 & 0.9 & 0.1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 0.81 & 0.18 & 0.01 \\ 0.1 & 0.81 & 0.09 \\ 0.9 & 0.1 & 0 \end{bmatrix}$$

$$P_{11}^4 = \begin{bmatrix} 0.81 & 0.18 & 0.01 \end{bmatrix}\begin{bmatrix} 0.81 \\ 0.1 \\ 0.9 \end{bmatrix}$$

$P_{11}^4 = \boxed{0.6831}$

ii) $\pi_1 = 0.9\pi_1 + \pi_3 \Rightarrow \pi_3 = 0.1\pi_1$

$, \pi_2 = 0.1\pi_1 + 0.9\pi_2 \Rightarrow \pi_2 = \pi_1$

$\&:- \pi_1 + \pi_2 + \pi_3 = 1 \Rightarrow \pi_1 = \frac{10}{21}$

$\Rightarrow \pi_2 = \frac{10}{21}, \pi_3 = \frac{1}{21}$

$\therefore$ limiting dist. is $\left(\frac{10}{21}, \frac{10}{21}, \frac{1}{21}\right)$

iii) Long run rate of repairs per unit of time $= \pi_3 = \boxed{\frac{1}{21}}$

b) Let $X(t)$ is the # of messages that arrive, $\lambda = 3$ (poisson process)

(i) $pr\{X(4) = 0\}$

$$= \frac{(\lambda t)^k e^{-\lambda t}}{k!} = \frac{(12)^0 e^{-12}}{0!} = e^{-12}$$

$t = 4$ min

8:00 AM

$\lambda t = 12$

(ii) exp. dist. with parameter $\lambda = 3$

$\Rightarrow$ p.d.f is $f(t) = 3e^{-3t}$

#

4

Q4 a) $X(0) = 1$ initial condition

$\Rightarrow P_1(0) = 1$

$$\Rightarrow P_n(0) = \begin{cases} 1 & , n = 1 \\ 0 & , \text{otherwise} \end{cases}$$

9

5+4

$\because \lambda_0 = 0$ ① $\Rightarrow \frac{dP_0(t)}{dt} = 0$

$\therefore$ $\boxed{P_0(t) = 0}$ (3) 1

$$② \Rightarrow \frac{dP_n(t)}{dt} = \lambda_{n-1} P_{n-1}(t) - \lambda_n P_n(t)$$

$$\frac{dP_n(t)}{dt} + \lambda_n P_n(t) = \lambda_{n-1} P_{n-1}(t)$$
$, n = 1, 2, \ldots$

$$\lambda_n = n\lambda, \lambda_{n-1} = (n-1)\lambda$$

$$\therefore \frac{dP_n(t)}{dt} + n\lambda P_n(t) = (n-1)\lambda P_{n-1}(t)$$
$, n = 1, 2, \ldots$

Multiply both sides by $e^{n\lambda t}$

$$e^{n\lambda t}\left[\frac{dP_n(t)}{dt} + n\lambda P_n(t)\right] = (n-1)\lambda P_{n-1}(t)\, e^{n\lambda t}$$

$$\therefore \frac{d}{dt}\left[P_n(t)\, e^{n\lambda t}\right] = (n-1)\lambda P_{n-1}(t)\, e^{n\lambda t}$$

$$\Rightarrow \int_0^t d\left[P_n(x)\, e^{n\lambda x}\right] = (n-1)\lambda \int_0^t P_{n-1}(x)\, e^{n\lambda x}\, dx$$

$$\therefore \left[P_n(x)\, e^{n\lambda x}\right]_0^t = (n-1)\lambda \int_0^t P_{n-1}(x)\, e^{n\lambda x}\, dx$$ 1

$$\Rightarrow \boxed{P_n(t) = e^{-n\lambda t}\left[P_n(0) + (n-1)\lambda \int_0^t P_{n-1}(x)\, e^{n\lambda x}\, dx\right]} \quad (4)$$
$n = 1, 2, \ldots$

which is a recurrence relation

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5 at $n=1$

$$P_1(t) = e^{-\lambda t}\,[\cancel{P_1(0)} + 0] = e^{-\lambda t} \qquad (5)$$

1

at $n=2$

$$P_2(t) = e^{-2\lambda t}\left[\cancel{P_2(0)} + \lambda \int_0^t P_1(x)\, e^{2\lambda x}\, dx\right]$$

$$(5) \Rightarrow P_1(x) = e^{-\lambda x}$$

$$P_2(t) = e^{-2\lambda t}\left[\lambda \int_0^t e^{-\lambda x}\, e^{2\lambda x}\, dx\right]$$

$$P_2(t) = \lambda\, e^{-2\lambda t} \int_0^t e^{\lambda x}\, dx$$

$$P_2(t) = e^{-\lambda t}(1 - e^{-\lambda t})^1$$

Similarly:

1

$$\boxed{P_n(t) = e^{-\lambda t}(1 - e^{-\lambda t})^{n-1}}$$

$$P_n(t) = p(1-p)^{n-1}, \quad p = e^{-\lambda t} \quad \text{(Geom. dist}^{n}\text{)}$$

$$n = 1, 2, \ldots$$

$$\therefore X(t) \sim \text{geom}(p), \quad p = e^{-\lambda t}$$

1/2

$$\text{Mean}[X(t)] = 1/p = e^{\lambda t}$$

1/2

$$\text{Variance}[X(t)] = \frac{1-p}{p^2} = \frac{1 - e^{-\lambda t}}{e^{-2\lambda t}}$$

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b) For Yule process

$$P_n(t) = e^{-\beta t}(1 - e^{-\beta t})^{n-1}, \quad n \geq 1$$

2

$$\Pr\{X(U) = k\} = \int_0^1 e^{-\beta u}(1 - e^{-\beta u})^{k-1}\, du$$

1

$$\Pr\{X(U) = k\} = \frac{1}{\beta k}\left[(1 - e^{-\beta})^k - 0\right]$$

1

$$= p^k / \beta k, \quad k = 1, 2, \ldots$$

where $p = 1 - e^{-\beta}$ #