King Saud University
College of Sciences
Department of Mathematics

Final Exam, S2 1439/1440
M 380 – Stochastic Processes
Time: 3 hours

Answer the following questions:

Q1: [8]

Suppose that the summands $\xi_1, \xi_2, \ldots$ are continuous random variables having a probability density function $f(z) = \begin{cases} \lambda e^{-\lambda z} & \text{for } z \geq 0 \\ 0 & \text{for } z < 0 \end{cases}$

and $P_N(n) = \beta(1-\beta)^{n-1}$ for $n = 1, 2, \ldots$

Find the probability density function for $X = \xi_1 + \xi_2 + \ldots + \xi_N$

Q2: [4+4]

Consider the Markov chain whose transition probability matrix is given by

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0.1 & 0.6 & 0.1 & 0.2 \\ 0.2 & 0.3 & 0.4 & 0.1 \\ 0 & 0 & 0 & 1 \end{array} \right\| \end{array}$$

(a) Starting in state 2, determine the probability that the Markov chain ends in state 0.

(b) Determine the mean time to absorption.

Q3: [4+4]

(a) Demands on a first aid facility in a certain location occur according to a nonhomogeneous Poisson process having the rate function

$$\lambda(t) = \begin{cases} 2t & \text{for } 0 \leq t < 1 \\ 2 & \text{for } 1 \leq t < 2 \\ 4 - t & \text{for } 2 \leq t \leq 4 \end{cases}$$

where t is measured in hours from the opening time of the facility. What is the probability that two demands occur in the first 2h of operation and two in the second 2h?

(b) Suppose that the social classes of successive generations in a family follow a Markov chain with transition probability matrix given by

		Son's class		
		Lower	Middle	Upper
Father's class	Lower	0.7	0.2	0.1
	Middle	0.2	0.6	0.2
	Upper	0.1	0.4	0.5

What fraction of families are upper class in the long run?

Q4: [4+5]

(a) Let X and Y be independent Poisson distributed random variables with parameters α and β, respectively. Determine the conditional distribution of X, given that $N = X + Y = n$.

(b) If $X(t)$ represents a size of a population where $X(0) = 1$, using the following differential equations

$$\frac{dp_0(t)}{dt} = -\lambda_0 p_0(t) \qquad (1)$$

$$\frac{dp_n(t)}{dt} = \lambda_{n-1} p_{n-1}(t) - \lambda_n p_n(t), \quad n=1,2,3, \ldots \qquad (2)$$

Prove that: $X(t) \sim geom\ (p)$, $p = e^{-\lambda t}$ when $\lambda_0 = 0$ and $\lambda_n = n\lambda$, and then find the mean and variance of this process.

Q5: [7]

A pure death process starting from $X(0) = 3$ has death parameters $\mu_0 = 0$, $\mu_1 = 3$, $\mu_2 = 2$ and $\mu_3 = 5$. Determine $P_n(t)$ for n=0, 1, 2, 3.

Model Answer

Q1: [8]

We have $f_X(z) = \sum_{n=1}^{\infty} f^n(z) P_N(n)$

$\because$ The $n-fold$ convolution of $f(z)$ is the Gamma density function, $n \geq 1$

$$\therefore \quad f^n(z) = \begin{cases} \dfrac{\lambda^n}{\Gamma(n)} z^{n-1} e^{-\lambda z} & z \geq 0 \\ 0 & z < 0 \end{cases}$$

$\Rightarrow$

$$f^n(z) = \begin{cases} \dfrac{\lambda^n}{(n-1)!} z^{n-1} e^{-\lambda z} & z \geq 0 \\ 0 & z < 0 \end{cases}$$

$$\therefore \; f_X(z) = \lambda \beta e^{-\lambda z} \sum_{n=1}^{\infty} \frac{[\lambda(1-\beta)z]^{n-1}}{(n-1)!}$$
$$= \lambda \beta e^{-\lambda z} . e^{\lambda(1-\beta)z}$$
$$= \lambda \beta e^{-\lambda \beta z}, \quad z \geq 0$$

$\therefore$ X has an exponential distribution with parameter $\lambda\beta$.

Q2: [4+4]

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0.1 & 0.6 & 0.1 & 0.2 \\ 0.2 & 0.3 & 0.4 & 0.1 \\ 0 & 0 & 0 & 1 \end{array} \right\| \end{array}$$

$u_i = pr\{X_T = 0 | X_0 = i\}$ for i=1,2,

$and \;\; v_i$ =E[$T | X_0 = i$] for i=1,2.

(a)

$$u_1 = p_{10} + p_{11}u_1 + p_{12}u_2$$
$$u_2 = p_{20} + p_{21}u_1 + p_{22}u_2$$

$\Rightarrow$

$$u_1 = 0.1 + 0.6u_1 + 0.1u_2$$
$$u_2 = 0.2 + 0.3u_1 + 0.4u_2$$

$\Rightarrow$

$$4u_1 - u_2 = 1 \qquad (1)$$
$$3u_1 - 6u_2 = -2 \qquad (2)$$

Solving (1) and (2), we get

$$u_1 = \tfrac{8}{21} \text{ and } u_2 = \tfrac{11}{21}$$

Starting in state 2, the probability that the Markov chain ends in state 0 is

$$u_2 = u_{20} = \tfrac{11}{21}$$

(b) Also, the mean time to absorption can be found as follows

$$v_1 = 1 + p_{11}v_1 + p_{12}v_2$$
$$v_2 = 1 + p_{21}v_1 + p_{22}v_2$$

$\Rightarrow$

$$v_1 = 1 + 0.6v_1 + 0.1v_2$$
$$v_2 = 1 + 0.3v_1 + 0.4v_2$$

$\Rightarrow$

$$4v_1 - v_2 = 10 \qquad (1)$$
$$3v_1 - 6v_2 = -10 \qquad (2)$$

Solving (1) and (2), we get $v_2 = v_{20} = \tfrac{10}{3}$

Q3: [4+4]

(a)

i)

$$\mu_1 = \int_0^2 \lambda(u)du$$
$$= \int_0^1 2t dt + \int_1^2 2dt$$
$$= \left[t^2\right]_0^1 + 2\left[t\right]_1^2$$
$$=3$$

The prob. that two demands occur in the first 2h of operation is

$$\Pr\{X(2)=2\} = \Pr\{X(2)-X(0)=2\}$$
$$= \frac{e^{-\mu_1}\mu_1^k}{k!}$$
$$= \frac{e^{-3}.3^2}{2!}$$
$$=0.2240$$

ii)

$$\mu_2 = \int_2^4 \lambda(u)du$$
$$= \int_2^4 (4-t)dt$$
$$= \left[4t\text{-}\frac{t^2}{2}\right]_2^4$$
$$=2$$

The prob. that two demands occur in the second 2h of operation is

$\Pr\{X(4) - X(2) = 2\}$

$$= \frac{e^{-\mu_2} \mu_2^k}{k!}$$

$$= \frac{e^{-2}.2^2}{2!}$$

$$=0.2707$$

(b)

Let $\pi = (\pi_0, \pi_1, \pi_2)$ be the limiting distribution

$\Rightarrow$

$\pi_0 = 0.7\pi_0 + 0.2\pi_1 + 0.1\pi_2$

$\pi_1 = 0.2\pi_0 + 0.6\pi_1 + 0.4\pi_2$

$\pi_2 = 0.1\pi_0 + 0.2\pi_1 + 0.5\pi_2$

$\pi_0 + \pi_1 + \pi_2 = 1$

Solving the following equations

$3\pi_0 - 2\pi_1 - \pi_2 = 0$ (1)

$\pi_0 + 2\pi_1 - 5\pi_2 = 0$ (2)

$\pi_0 + \pi_1 + \pi_2 = 1$ (3)

We get $\pi_0 = \frac{6}{17}$, $\pi_1 = \frac{7}{17}$, $\pi_2 = \frac{4}{17}$

In the long run, approximately 23.5% of families are upper class.

Q4: [4+5]

(a)

$$X \sim Poisson(\alpha), \quad Y \sim Poisson(\beta)$$

$$\therefore \text{ X+Y} \sim Poisson(\alpha+\beta)$$

$$\Rightarrow$$

$$\Pr\{X=k|N=n\} = \Pr\{X=k|X+Y=n\}$$

$$= \frac{\Pr\{X=k\}\bigcap\Pr\{X+Y=n\}}{\Pr\{X+Y=n\}}$$

$$= \frac{\Pr\{X=k\}\bigcap\Pr\{Y=n-k\}}{\Pr\{X+Y=n\}}$$

$$= \frac{e^{-\alpha}\alpha^k/k!\ .\ e^{-\beta}\beta^{n-k}/(n-k)!}{e^{-(\alpha+\beta)}(\alpha+\beta)^n/n!}$$

$$= \alpha^k\beta^{n-k}\left(\frac{1}{\alpha+\beta}\right)^n \frac{n!}{k!(n-k)!}$$

$$= \binom{n}{k}\left(\frac{\alpha}{\alpha+\beta}\right)^k\left(\frac{\beta}{\alpha+\beta}\right)^{n-k}$$

$$= \binom{n}{k}p^k(1-p)^{n-k} \qquad \text{Binomial distribution}$$

where $p=\frac{\alpha}{\alpha+\beta}$ $and\ 1-p=\frac{\beta}{\alpha+\beta}$

(b)

$$\frac{dp_0(t)}{dt} = -\lambda_0 p_0(t) \qquad (1)$$

$$\frac{dp_n(t)}{dt} = \lambda_{n-1}p_{n-1}(t) - \lambda_n p_n(t), \ n=1,2,3, \dots \qquad (2)$$

The initial condition is $X(0)=1 \ \Rightarrow \ p_1(0)=1$

$$\Rightarrow \ p_n(0) = \begin{cases} 1 & \text{, n=1} \\ 0 & \text{, otherwise} \end{cases}$$

$$\lambda_0 = 0 \qquad (1) \Rightarrow \frac{dp_0(t)}{dt} = 0$$

$$\Rightarrow \ p_0(t) = 0 \qquad (3)$$

$$(2) \Rightarrow \ \frac{dp_n(t)}{dt} = \lambda_{n-1} p_{n-1}(t) - \lambda_n p_n(t)$$

$$\Rightarrow \ \frac{dp_n(t)}{dt} + \lambda_n p_n(t) = \lambda_{n-1} p_{n-1}(t), \ n = 1, 2, \ \ldots$$

$$\because \ \lambda_n = n\lambda, \ \ \lambda_{n-1} = (n-1)\lambda$$

$$\therefore \ \frac{dp_n(t)}{dt} + n\lambda p_n(t) = (n-1)\lambda p_{n-1}(t), \ \text{n=1,2, ...}$$

Multiply both sides by $e^{n\lambda t}$

$$e^{n\lambda t}\left[\frac{dp_n(t)}{dt} + n\lambda p_n(t)\right] = (n-1)\lambda p_{n-1}(t) e^{n\lambda t}$$

$$\therefore \ \frac{d}{dt}\left[p_n(t) e^{n\lambda t}\right] = (n-1)\lambda p_{n-1}(t) e^{n\lambda t}$$

$$\Rightarrow \int_0^t d\left[p_n(x) e^{n\lambda x}\right] = (n-1)\lambda \int_0^t p_{n-1}(x) e^{n\lambda x} dx$$

$$\therefore \ \left[p_n(x) e^{n\lambda x}\right]_0^t = (n-1)\lambda \int_0^t p_{n-1}(x) e^{n\lambda x} dx$$

$$\Rightarrow \ p_n(t) = e^{-n\lambda t}\left[p_n(0) + (n-1)\lambda \int_0^t p_{n-1}(x) e^{n\lambda x} dx\right], \ n = 1, 2, \ \ldots \quad (4)$$

which is a recurrence relation.

at $n = 1$

$$p_1(t) = e^{-\lambda t}\left[p_1(0) + 0\right] = e^{-\lambda t} \qquad (5)$$

at $n = 2$

$$p_2(t) = e^{-2\lambda t}\left[p_2(0) + \lambda\int_0^t p_1(x)e^{2\lambda x}dx\right]$$

$(5) \Rightarrow \ p_1(x) = e^{-\lambda x}$

$$\therefore \ p_2(t) = e^{-2\lambda t}\left[\lambda\int_0^t e^{-\lambda x}e^{2\lambda x}dx\right]$$

$$\therefore \ p_2(t) = \lambda e^{-2\lambda t}\int_0^t e^{\lambda x}dx$$

$$= e^{-\lambda t}(1-e^{-\lambda t})^1 \qquad (6)$$

Similarly as (5) and (6), we deduce that

$$p_n(t) = e^{-\lambda t}(1-e^{-\lambda t})^{n-1}$$

$$= p(1-p)^{n-1}, \quad p = e^{-\lambda t}, \ n = 1, 2, \ \ldots$$

$\therefore X(t) \sim geom(p), \ p = e^{-\lambda t}$

$Mean\left[X(t)\right] = 1/p = e^{\lambda t}$,

$$Variance\left[X(t)\right] = \frac{1-p}{p^2} = \frac{1-e^{-\lambda t}}{e^{-2\lambda t}}$$

Q5: [7]

The transition probabilities are given by

$$p_N(t) = e^{-\mu_N t} \qquad (1)$$

and for $n < N$

$$p_n(t) = pr\left\{X(t) = n \middle| X(0) = N\right\}$$

$$= \mu_{n+1}\mu_{n+2} \ \ldots \ \mu_N\left[A_{n,n}e^{-\mu_n t} + \ \ldots \ + A_{k,n}e^{-\mu_k t} + \ \ldots \ + A_{N,n}e^{-\mu_N t}\right] \quad (2)$$

Where $A_{k,n} = \prod_{i=N}^{n}\frac{1}{(\mu_i - \mu_k)}$, $\ i \neq k, \ n \leq k \leq N, \ i = N, N\text{-}1, \ \ldots, \ n \quad (3)$

For N=3 $\quad (1) \Rightarrow \text{p}_3\text{(t)=e}^{-\mu_3 t}$

$\therefore \ \text{p}_3\text{(t)=e}^{-5t} \qquad$ (I)

For n=2 $\quad (2) \Rightarrow \text{p}_2\text{(t)=}\mu_3\left[A_{2,2}e^{-\mu_2 t} + A_{3,2}e^{-\mu_3 t}\right]$

$$(3) \Rightarrow \ A_{2,2} = \prod_{i=3}^{2}\frac{1}{(\mu_i - \mu_2)} \ , \ i \neq 2$$

$$= \frac{1}{\mu_3 - \mu_2} = \frac{1}{3}$$

$$, A_{3,2}=\prod_{i=3}^{2}\frac{1}{(\mu_i - \mu_3)},\ i\neq 3$$

$$=\frac{1}{\mu_2-\mu_3}=\frac{-1}{3}$$

$$\therefore\ p_2(t)=5\left[\frac{1}{3}e^{-2t}-\frac{1}{3}e^{-5t}\right] \qquad \text{(II)}$$

For n=1 $(2)\Rightarrow p_1(t)=\mu_2\mu_3\left[A_{1,1}e^{-\mu_1 t}+A_{2,1}e^{-\mu_2 t}+A_{3,1}e^{-\mu_3 t}\right]$

$$(3)\Rightarrow\ A_{1,1}=\prod_{i=3}^{1}\frac{1}{(\mu_i-\mu_1)},\ i\neq 1$$

$$=\frac{1}{(\mu_3-\mu_1)(\mu_2-\mu_1)}=-\frac{1}{2}$$

$$A_{2,1}=\prod_{i=3}^{1}\frac{1}{(\mu_i-\mu_2)},\ i\neq 2$$

$$=\frac{1}{(\mu_3-\mu_2)(\mu_1-\mu_2)}=\frac{1}{3}$$

$$, A_{3,1}=\prod_{i=3}^{1}\frac{1}{(\mu_i-\mu_3)},\ i\neq 3$$

$$=\frac{1}{(\mu_2 - \mu_3)(\mu_1 - \mu_3)}=\frac{1}{6}$$

$$\therefore\ p_1(t)=10\left[-\frac{1}{2}e^{-3t}+\frac{1}{3}e^{-2t}+\frac{1}{6}e^{-5t}\right] \qquad \text{(III)}$$

Using (I), (II) and (III) we can get $p_0(t)$ as follows

$$\therefore\ p_0(t)=1-\left[p_1(t)+p_2(t)+p_3(t)\right]$$

$$=1-\left[-5e^{-3t}+\frac{10}{3}e^{-2t}+\frac{5}{3}e^{-5t}+\frac{5}{3}e^{-2t}-\frac{5}{3}e^{-5t}+e^{-5t}\right]$$

$$=1+5e^{-3t}-5e^{-2t}-e^{-5t} \qquad \text{(IV)}$$