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Solutions to Exercises on Random Variables and their distributions

1. An urn contains N bulls numbered from 1 to N. We pick a randomly a bull (all the bulls are equally likely to be extracted) and define the r.v. X by the number of the extracted bull.

   (a) Calculate the expectation and the variance of X.
   Remark fist that X is a uniform discrete random variable on $\{1, 2, 3, \ldots, N\}$.

$$E[X] = \sum_{k=1}^{N} kP(X = k) = \frac{1}{N}\sum_{k=1}^{N} k = \frac{(1+N)N}{2N} = \frac{1+N}{2}$$

   and

$$E\left[X^2\right] = \sum_{k=1}^{N} k^2 P(X = k) = \frac{1}{N}\sum_{k=1}^{N} k^2 = \frac{N(2N+1)(N+1)}{N6}.$$

   Hence

$$Var(X) = E\left[X^2\right] - (E[X])^2 = \frac{(2N+1)(N+1)}{6} - \frac{(1+N)^2}{4} = \frac{N^2-1}{12}.$$

2. Let X be a r.v. with values in $\mathbb{N}$ such that: $\forall\, n \in \mathbb{N}^*$, $P(X = n) = \dfrac{4}{n}P(X = n-1)$.

   (a) Find $P(X = 0)$
   We have

$$\begin{aligned} P(X = n) &= \frac{4}{n}P(X = n-1) = \frac{4}{n}\frac{4}{n-1}\cdots\frac{4}{2}\frac{4}{1}P(X = 0) \\ &= \frac{4^n}{n!}P(X = 0) \quad \forall\, n \in \mathbb{N}^*. \end{aligned}$$

   We know that

$$\begin{aligned} \sum_{n=0}^{\infty} P(X = n) &= 1 \Longleftrightarrow P(X = 0) + \sum_{n=1}^{\infty} \frac{4^n}{n!}P(X = 0) = 1 \\ &\Longleftrightarrow P(X = 0)\left(\sum_{n=0}^{\infty} \frac{4^n}{n!}\right) = 1 \\ &\Longleftrightarrow P(X = 0) = e^{-4} \end{aligned}$$

   (b) Find the distribution of X and calculate its expectation and its variance.

3. Two players are tossing fair coins. A tosses $(n+1)$ times the coin and B tosses n times the coin ($n \in \mathbb{N}^*$). Let X and Y be the number of "heads" got respectively by the player A and the player B.

   (a) Calculate the probability of the following events $\{X - Y = k\}$, $k \in \mathbb{Z}$, $\{X = Y\}$, $\{X > Y\}$.

4. We toss n times a fair coin and define the r.v. X to be the number of tails got after n tosses and define the r.v. $Y = \dfrac{a^X}{2^n}$, $(a \in \mathbb{R}_+^*)$. Calculate $E[Y]$.

5. Let X be a Poison r.v. with parameter λ and define the r.v. Y by

$$Y = \begin{cases} \frac{X}{2} & \text{if } X \text{ is even} \\ 0 & \text{if } X \text{ is odd.} \end{cases}$$

   (a) Find the distribution Y, and calculate its expectation and its variance.

6. Let X and Y be two independent r.v. taking values in $\mathbb{N}$: such that X follows the Bernoulli distribution with parameter p and Y follows a Poisson distribution of parameter λ. Now define the r.v. Z by $Z = XY$.

   (a) Calculate the distribution of Z.
   (b) Find the moment generating function (MGF) of Z.
   (c) Deduce $E[Z]$ and $Var[Z]$.
   (d) Calculate $P(X = 1 \mid Z = 0)$.

7. Let X_1 and X_2 be two i.i.d. r.v. with values in $\mathbb{N}$ such that :

$$\forall\, k \in \mathbb{N},\ P(X_1 = k) = \frac{1}{2^{k+1}}.$$

   (a) Find the distribution and calculate the expectation of $Y = \max(X_1, X_2)$.
   Solution. i) $Y(\Omega) = \mathbb{N}$, ii) $\forall\, k \in \mathbb{N}\ p(k) = P(Y = k)$.
   We have $Y(\Omega) = \mathbb{N}$, ii) $\forall\, k \in \mathbb{N}\ p(k) = P(Y = k)$. We have

$$\begin{aligned} P(Y = k) &= P(\max(X_1, X_2) = k) \\ &= P(X_2 = k, X_1 < X_2) + P(X_1 = k, X_1 \geq X_2) \\ &= P(X_2 = k, X_1 < k) + P(X_1 = k, X_2 \leq k) \\ &= P(X_2 = k)\, P(X_1 < k) + P(X_1 = k)\, P(X_2 \leq k) \\ &= P(X_2 = k)\,[P(X_1 < k) + P(X_2 \leq k)] \ \ (\text{since } P(X_1 = k) = P(X_2 = k)\,) \\ &= P(X_1 = k)\,[2P(X_1 \leq k) - P(X_1 = k)] \\ &= \frac{1}{2^{k+1}} \left(\sum_{i=0}^{k} \frac{2}{2^{i+1}} - \frac{1}{2^{k+1}} \right) = \frac{1}{2^{k+1}} \left(\frac{1 - 2^{-(k+1)}}{1 - 2^{-1}} \right) - \frac{1}{2^{2k+2}} \\ &= \frac{1}{2^k} \left(1 - \frac{1}{2^{k+1}} \right) - \frac{1}{2^{2k+2}} = \frac{1}{2^k} - \frac{3}{2^{2k+2}} \end{aligned}$$

   where we have used the relation $P(X_1 < k) = P(X_1 \leq k) - P(X_1 = k))$

8. Let X and Y be two r.v. taking values in $\mathbb{N}$ such that:

$$\forall\, m \in \mathbb{N}^* \text{ and } \ \forall\, n \in \mathbb{N} \ \ P(\{X = m\} \cap \{Y = n\}) = \frac{e^{-1}}{n!} \times \frac{1}{2^m}.$$

   (a) Find the distributions of X and Y.
   i) $X(\Omega) = \mathbb{N}^*$ and $Y(\Omega) = \mathbb{N}$, ii) $\forall\, m \in \mathbb{N}^*$ we have

$$\begin{aligned} P(X = m) &= \sum_{n=0}^{\infty} P(X = m, Y = n) \\ &= \sum_{n=0}^{\infty} \frac{e^{-1}}{n!} \frac{1}{2^m} = \frac{1}{2^m} \quad (\text{because } \sum_{n=0}^{\infty} \frac{e^{-1}}{n!} = 1). \end{aligned}$$

Hence $X \hookrightarrow \mathcal{G}(\frac{1}{2})$. And

$$\begin{aligned} P(Y=n) &= \sum_{m=1}^{\infty} P(X=m, Y=n) \\ &= \sum_{m=1}^{\infty} \frac{e^{-1}}{n!}\frac{1}{2^m} = \frac{e^{-1}}{n!} \quad \text{(because } \sum_{m=1}^{\infty}\frac{1}{2^m} = 1\text{).} \end{aligned}$$

Then $Y \hookrightarrow \mathcal{P}(1)$.

(b) Are X and Y independent ?
We have $\forall\, m \in \mathbb{N}^*$ and $\forall\, n \in \mathbb{N}$

$$P(X=m, Y=n) = P(X=m)\,P(Y=n)$$

therefore X and Y are independent.

(c) Find their expectation and their variance.

$$E[X] = \frac{1}{\frac{1}{2}} = 2 \quad \text{and} \quad Var(X) = \frac{1-\frac{1}{2}}{\left(\frac{1}{2}\right)^2} = 2$$

and

$$E[Y] = 1 \ \text{ and } \ Var(Y) = 1$$

9. Let f be a function defined by:

$$f(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } 0 \le x < 1. \\ 0 & \text{if } x \ge 1 \end{cases}$$

(a) show that f is a probability density function of a r.v. X.
f is a p.d.f. because $f(x) \ge 0$ foe all $x \in \mathbb{R}$, and $\int_{\mathbb{R}} f(x)dx = \int_0^1 1dx = 1$

(b) Find the c.d.f. of X.
The c.d.f. is given by

$$F_X(x) = \int_{-\infty}^{x} f(t)dt = \begin{cases} 0 & \text{if} \quad x \le 0 \\ \int_0^x 1dt & \text{if} \quad 0 < x < 1 \\ 1 & \text{if} \quad x \ge 1. \end{cases} = \begin{cases} 0 & \text{if} \quad x \le 0 \\ x & \text{if} \quad 0 < x < 1 \\ 1 & \text{if} \quad x \ge 1. \end{cases}$$

(c) Calculate $E[X]$ and its variance.
We have

$$E[X] = \int_{-\infty}^{+\infty} tf(t)dt = \int_0^1 tdt = \frac{1}{2}$$

and

$$E\left[X^2\right] = \int_{-\infty}^{+\infty} t^2 f(t)dt = \int_0^1 t^2 dt = \frac{1}{3}$$

Then

$$Var(X) = E\left[X^2\right] - (E[X])^2 = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{12}$$

10. Let X be r.v. having a p.d.f. f given by:

$$f(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ a(x+1) & \text{if} \quad 0 \le x < 2 \\ a(x-1) & \text{if} \quad 2 \le x < 4 \\ 0 & \text{if} \quad x \ge 4 \end{cases}$$

(a) Find the value of the constant a

$$\int_0^2 f(x)dx = 1 \Longleftrightarrow \int_0^2 a(x+1)dx + \int_2^4 a(x-1)dx = 1 \Longleftrightarrow 8a = 1$$

hence $a = \frac{1}{8}$

(b) Give the c.d.f F_X of X

$$F_X(x) = \int_{-\infty}^{x} f(t)dt = \begin{cases} 0 & \text{if} \quad x < 0 \\ \int_0^x \frac{1}{8}(t+1)dt & \text{if} \quad 0 \le x < 2 \\ \int_0^2 \frac{1}{8}(t+1)dt + \int_2^x \frac{1}{8}(t-1)dt & \text{if} \quad 2 \le x < 4 \\ 1 & \text{if} \quad x \ge 4 \end{cases}$$

but we have

$$\int_0^x \frac{1}{8}(t+1)dt = \frac{1}{16}x\,(x+2)$$

and

$$\int_0^2 \frac{1}{8}(t+1)dt + \int_2^x \frac{1}{8}(t-1)dt = \frac{1}{16}x\,(x-2) + \frac{1}{2}$$

Therefore

$$= \begin{cases} 0 & \text{if} \quad x < 0 \\ \frac{1}{16}x\,(x+2) & \text{if} \quad 0 \le x < 2 \\ \frac{1}{16}x\,(x-2) + \frac{1}{2} & \text{if} \quad 2 \le x < 4 \\ 1 & \text{if} \quad x \ge 4 \end{cases}$$

(c) Deduce the value $P\,[1 \le X < 3]$.
We have

$$\begin{aligned} P\,[1 \le X < 3] &= F_X(3) - F_X(1) \\ &= \frac{1}{16}3\,(3-2) + \frac{1}{2} - \frac{1}{16}1\,(1+2) = \frac{1}{2} \end{aligned}$$

(d) Calculate $E\,[X]$ and $Var\,[X]$
We have

$$E\,[X] = \int_{-\infty}^{+\infty} tf(t)dt = \int_0^2 t\frac{1}{8}(t+1)dt + \int_2^4 t\frac{1}{8}(t-1)dt = \frac{13}{6}$$

and

$$E\left[X^2\right] = \int_{-\infty}^{+\infty} t^2 f(t)dt = \int_0^2 t^2\frac{1}{8}(t+1)dt + \int_2^4 t^2\frac{1}{8}(t-1)dt = 6$$

Then

$$Var(X) = E\left[X^2\right] - (E\,[X])^2 = 6 - \left(\frac{13}{6}\right)^2 = \frac{47}{36}$$

11. Let X be a continuous r.v. with p.d.f. f such that

$$f(x) = \begin{cases} c\ln x & \text{if} \quad 0 < x < 1 \\ 0 & \text{if} \quad \text{otherwise} \end{cases}$$

(a) Find the true value of c.
$\int_0^1 c\ln(x)\,dx = [c(x\ln(x) - x)]_0^1 = -c = 1$ hence $c = -1$

(b) Give the c.d.f F_X of X

$$F_X(x) = \int_{-\infty}^{x} f(t)dt = \begin{cases} 0 & \text{if} \quad x \leq 0 \\ \int_0^x -\ln(t)dt & \text{if} \quad 0 < x < 1 \\ 1 & \text{if} \quad x \geq 1. \end{cases}$$

Hence the c.d.f. is given by

$$F_X(x) = \begin{cases} 0 & \text{if} \quad x \leq 0 \\ x - x\ln(x) & \text{if} \quad 0 < x < 1 \\ 1 & \text{if} \quad x \geq 1. \end{cases}$$

(c) Calculate $E[X]$ and $Var[X]$

$$E[X] = \int_0^1 -x\ln(x)\,dx = \left[\frac{1}{4}x^2 - \frac{1}{2}x^2\ln x\right]_0^1 = \frac{1}{4}$$

and

$$E\left[X^2\right] = \int_0^1 -x^2\ln(x)\,dx = \left[\frac{1}{9}x^3 - \frac{1}{3}x^3\ln x\right]_0^1 = \frac{1}{9}$$

and then

$$Var[X] = \frac{1}{9} - \frac{1}{16} = \frac{7}{144}$$

(d) Let X be a continuous r.v. with c.d.f. F given by

$$F(x) = \begin{cases} 0 & \text{if} \quad x \leq x_0 \\ 1 - \dfrac{K}{x^2} & \text{if} \quad x_0 < x < +\infty \end{cases}$$

(e) Find the p.d.f. f of X and find the value of K.
The p.d.f. f of X is given $f(x) = F'(x) = \dfrac{2K}{x^3}$. To find K we solve the equation

$$\int_{x_0}^{+\infty} f(x)dx = 1 \Longleftrightarrow K\int_{x_0}^{+\infty} \frac{2}{x^3}dx = 1 \Longleftrightarrow \frac{K}{x_0^2} = 1 \Longleftrightarrow K = x_0^2$$

Hence $f(x) = \dfrac{2x_0^2}{x^3}$

12. Let X be standard normal r.v. that is $X \hookrightarrow \mathcal{N}(0,1)$.

(a) Find the distribution of the r.v. $Y = \frac{X^2}{2}$.
We have $Y(\Omega) = \mathbb{R}^+$ then $F_Y(y) = 0$ and $f_Y(y) = 0$, for $y \leq 0$. For $y > 0$,

$$\begin{aligned} F_Y(y) &= P(X^2 \leq 2y) \\ &= P\left(-\sqrt{2y} \leq X \leq \sqrt{2y}\right) \\ &= F_X\left(\sqrt{2y}\right) - F_X\left(-\sqrt{2y}\right). \end{aligned}$$

Hence

$$f_Y(y) = F_Y'(y) = \frac{F_X'\left(\sqrt{2y}\right) + F_X'\left(-\sqrt{2y}\right)}{\sqrt{2y}}$$

But remember that

$$F_X'(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right).$$

hence

$$f_Y(y) = \frac{1}{\sqrt{4y\pi}}\left(\exp(-y) + \exp(-y)\right) = \frac{\exp(-y)}{\sqrt{y\pi}} \text{ for all } y > 0,$$

(b) Deduce $E[X^2]$ and $Var[X^2]$.

$$E[X^2] = E[2Y] = 2\int_0^{\infty} \frac{ye^{-y}}{\sqrt{y\pi}}dy = \frac{2}{\sqrt{\pi}}\int_0^{\infty} \sqrt{y}e^{-y}dy = \frac{2}{\sqrt{\pi}}\Gamma\left(\frac{3}{2}\right)$$

where $\Gamma(p) = \int_0^{+\infty} e^{-t}t^{p-1}dt$.
And

$$E[X^4] = E[4Y^2] = 4\int_0^{\infty} \frac{y^2e^{-y}}{\sqrt{y\pi}}dy = \frac{4}{\sqrt{\pi}}\int_0^{\infty} y^{\frac{5}{2}}e^{-y}dy = \frac{4}{\sqrt{\pi}}\Gamma\left(\frac{7}{2}\right).$$

Consequently

$$Var[X^2] = E[X^4] - \left(E[X^2]\right)^2 = \frac{4}{\sqrt{\pi}}\left(\Gamma\left(\frac{7}{2}\right) - \frac{1}{\pi}\Gamma\left(\frac{3}{2}\right)^2\right).$$

13. Find the MGF of the following distributions and deduce their expectation and their variance

 (a) 1. Bernoulli distribution, 2. Binomial distribution, 3. Poisson distribution, 4. Geometric distribution, 5. Normal distribution with mean μ and variance σ^2.

14. Let X be a r.v. taking values in $\{-b, -a, a, b\}$ (where a and b are real numbers such that $0 < a < b$). Set $Y = X^2$.

 (a) Find the distribution of Y and the distribution of the couple (X, Y).
 We have $Y(\Omega) = \{a^2, b^2\}$ and

$$\begin{aligned} P\left(Y = a^2\right) &= P\left(X = a \text{ or } X = -a\right) \\ &= P\left(X = -a\right) + P\left(X = a\right) \\ &= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}. \end{aligned}$$

 Consequently

$$P\left(Y = b^2\right) = 1 - P\left(Y = a^2\right) = \frac{1}{2}.$$

 We have $(X, Y)(\Omega) = (X(\Omega), Y(\Omega)) = \{(-b, b^2); (b, b^2); (-a, a^2); (a, a^2)\}$ and

$$\begin{aligned} P\left((X,Y) = (-b, b^2)\right) &= P\left(X = -b, Y = b^2\right) = P\left(X = -b\right) = \frac{1}{4}, \\ P\left((X,Y) = (b, b^2)\right) &= P\left(X = b, Y = b^2\right) = P\left(X = b\right) = \frac{1}{4} \\ P\left((X,Y) = (-a, a^2)\right) &= P\left(X = -a, Y = a^2\right) = P\left(X = -a\right) = \frac{1}{4} \\ P\left((X,Y) = (a, a^2)\right) &= P\left(X = a, Y = a^2\right) = P\left(X = a\right) = \frac{1}{4} \end{aligned}$$

 (b) Show that $Covar(X, Y) = 0$.
 By definition we have

$$\begin{aligned} Covar(X, Y) &= E\left[XY\right] - E\left[X\right]E\left[Y\right] \\ &= E\left[X^3\right] - E\left[X\right]E\left[X^2\right] \end{aligned}$$

 But

$$E\left[X\right] = \frac{1}{4}\left(-b - a + a + b\right) = 0 \text{ and } E\left[X^3\right] = \frac{1}{4}\left(-b^3 - a^3 + a^3 + b^3\right) = 0$$

 Therefore $Covar(X, Y) = 0$.

(c) Are X and Y independent. We have $Y = X^2$ (Y is a function of X) hence they are dependent.

15. Let X_1 and X_2 be two independent r.v. such that:

$$X_1(\Omega) = X_2(\Omega) = \{-1, 1\} \text{ and } P(\{X_1 = 1\}) = P(\{X_2 = 1\}) = \frac{1}{2}$$

(a) Set $X_3 = X_1 X_2$. Are the r.v. X_1, X_2 and X_3 mutually independent ?
We calculate

$$\begin{aligned} P(X_1 = 1, X_2 = -1, X_3 = -1) &= P(X_1 = 1, X_2 = -1) \\ &= P(X_1 = 1) P(X_2 = -1) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}. \end{aligned}$$

On the other hand

$$P(X_1 = 1) P(X_2 = -1) P(X_3 = -1) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$$

because $X_3(\Omega) = \{-1, 1\}$

$$\begin{aligned} P(X_3 = 1) &= P(X_1 = -1, X_2 = -1) + P(X_1 = 1, X_2 = 1) \\ &= \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{1}{2}. \end{aligned}$$

Therefore

$$P(X_1 = 1, X_2 = -1, X_3 = -1) \neq P(X_1 = 1) P(X_2 = -1) P(X_3 = -1).$$

X_1, X_2 and X_3 are not mutually independent.

16. Let X and Y be two r.v. with the following distribution:

$$X(\Omega) = \{-1, 1\} \text{ such that } P(X = -1) = \frac{1}{4} \text{ and } Y(\Omega) = \{1, 2\} \text{ such that } P(Y = 1) = \frac{1}{3}$$

Denote by p the probability of the event $\{X = -1\} \cap \{Y = 1\}$.

(a) Find the joint probability distribution of the couple (X, Y) in terms of p.
We have $(X, Y)(\Omega) = (X(\Omega), Y(\Omega)) = \{(-1, 1); (-1, 2); (1, 1); (1, 2)\}$. Moreover we know that

$$P(X = -1, Y = 1) + P(X = -1, Y = 2) = P(X = -1) = \frac{1}{4}$$

and

$$P(X = -1, Y = 1) + P(X = 1, Y = 1) = P(Y = 1) = \frac{1}{3}.$$

and

$$P(X = -1, Y = 2) + P(X = 1, Y = 2) = P(Y = 2) = \frac{2}{3}$$

Then

$$P(X = -1, Y = 1) = p \quad \text{and} \quad P(X = -1, Y = 2) = \frac{1}{4} - p$$

and

$$P(X = 1, Y = 1) = \frac{1}{3} - p \quad \text{and} \quad P(X = 1, Y = 2) = \frac{2}{3} - \frac{1}{4} + p = \frac{5}{12} + p.$$

(b) What the required conditions of p?
The parameter p should satisfy $0 < p < 1$, $0 < \frac{1}{4} - p < 1$, $0 < \frac{1}{3} - p < 1$ and $\frac{5}{12} + p < 1$ that is $0 < p < \frac{1}{4}$

(c) Find the values of p in such away that X and Y become independent.
If X and Y are independent then we should have

$$P(X = 1, Y = 1) = \frac{1}{3} - p = P(X = 1) P(Y = 1) = \frac{3}{4}\frac{1}{3} = \frac{1}{4}$$

which implies that $p = \frac{1}{12}$. And

$$P(X = -1, Y = 2) = \frac{1}{4} - p = P(X = -1) P(Y = 2) = \frac{1}{4}\frac{2}{3} = \frac{1}{6}$$

which implies that $p = \frac{1}{12}$. And

$$P(X = -1, Y = 1) = p = P(X = -1) P(Y = 1) = \frac{1}{4}\frac{1}{3} = \frac{1}{12}.$$

And

$$P(X = 1, Y = 2) = \frac{5}{12} + p = P(X = 1) P(Y = 2) = \frac{3}{4}\frac{2}{3} = \frac{1}{2}$$

which implies that $p = \frac{1}{12}$. So X and Y are independent is and only if $p = \frac{1}{12}$.

(d) Find in this case the distributions of the following r.v.:

$$Z = XY\ ; \quad S = X + Y\ ; \quad D = X - Y\ ; \quad M = \max(X, Y)\ ; \quad I = \min(X, Y).$$

Solutions.

i. We have $p = 12$, i. $Z(\Omega) = \{-2, -1, 1, 2\}$ and ii. Probability mass function:

$$\begin{aligned} P(Z = -2) &= P(X = -1, Y = 2) = \frac{1}{6}, \quad P(Z = -1) = P(X = -1, Y = 1) = \frac{1}{12} \\ P(Z = 2) &= P(X = 1, Y = 2) = \frac{1}{2}, \quad P(Z = 1) = P(X = 1, Y = 1) = \frac{1}{4} \end{aligned}$$

ii. We have i. $S(\Omega) = \{0, 1, 2, 3\}$ and ii. Probability mass function:

$$\begin{aligned} P(S = 0) &= P(X = -1, Y = 1) = \frac{1}{12}, \quad P(S = 1) = P(X = -1, Y = 2) = \frac{1}{6} \\ P(S = 2) &= P(X = 1, Y = 1) = \frac{1}{4}, \quad P(S = 3) = P(X = 1, Y = 2) = \frac{1}{2} \end{aligned}$$

iii. We have i. $D(\Omega) = \{-3, -2, -1, 0, 1\}$ and ii. Probability mass function:

$$\begin{aligned} P(D = -3) &= P(X = -1, Y = 2) = \frac{1}{6}, \ P(D = -2) = P(X = -1, Y = 1) = \frac{1}{12}, \\ P(D = -1) &= P(X = 1, Y = 2) = \frac{1}{2}, \ P(D = 0) = P(X = 1, Y = 1) = \frac{1}{4}, \end{aligned}$$

iv. We have i. $M(\Omega) = \{1, 2\}$ and ii. Probability mass function:

$$\begin{aligned} P(M = 1) &= P(X = -1, Y = 1 \text{ or } X = 1, Y = 1) \\ &= P(X = -1, Y = 1) + P(X = 1, Y = 1) \\ &= \frac{1}{12} + \frac{1}{4} = \frac{1}{3} \\ P(M = 2) &= 1 - \frac{1}{3} = \frac{2}{3} \end{aligned}$$

v. We have i. $I(\Omega) = \{-1, 1\}$ and ii. Probability mass function:

$$\begin{aligned} P(I=1) &= P(X=1, Y=1 \text{ or } X=1, Y=2) \\ &= P(X=1, Y=1) + P(X=1, Y=2) \\ &= \frac{1}{4} + \frac{1}{2} = \frac{3}{4} \\ P(I=-1) &= 1 - \frac{3}{4} = \frac{1}{4}. \end{aligned}$$

17. Let X and Z be two r.v. with integer values. Assume that Z is a Poisson r.v. with parameter λ such that

$$X \leq Z \quad \text{and} \quad \forall\, n \geq 0, \quad \forall\, k \leq n, \quad P(X = k/Z = n) = C_n^k p^k (1-p)^{n-k} \ (0 < p < 1).$$

(a) Show that X and $Y = Z - X$ are two independent Poisson r.v.
We can write

$$\begin{aligned} P(X=k, Y=j) &= P(X=k, Z=j+k) \\ &= P(X=k/Z=j+k)\, P(Z=j+k) \\ &= C_{j+k}^k p^k (1-p)^j e^{-\lambda} \frac{\lambda^{j+k}}{(j+k)!} \\ &= \frac{(p\lambda)^k}{k!} \frac{((1-p)\lambda)^j}{j!} e^{-\lambda} \\ &= e^{-\lambda p} \frac{(p\lambda)^k}{k!} \cdot e^{-\lambda(1-p)} \frac{((1-p)\lambda)^j}{j!}. \end{aligned}$$

Now, by taking the summation over j we get

$$\begin{aligned} P(X=k) &= \sum_{j=0}^{\infty} P(X=k, Y=j) \\ &= e^{-\lambda p} \frac{(p\lambda)^k}{k!} \cdot e^{-\lambda(1-p)} \sum_{j=0}^{\infty} \frac{((1-p)\lambda)^j}{j!} \\ &= e^{-\lambda p} \frac{(p\lambda)^k}{k!} \end{aligned}$$

and by taking the summation over k we get

$$\begin{aligned} P(Y=j) &= \sum_{k=0}^{\infty} P(X=k, Y=j) \\ &= e^{-\lambda(1-p)} \frac{((1-p)\lambda)^j}{j!} e^{-\lambda p} \sum_{j=0}^{\infty} \frac{(p\lambda)^k}{k!} \\ &= e^{-\lambda(1-p)} \frac{((1-p)\lambda)^j}{j!}. \end{aligned}$$

Consequently $X \hookrightarrow \mathcal{P}(p\lambda)$, $Y \hookrightarrow \mathcal{P}((1-p)\lambda)$ and

$$P(X=k, Y=j) = P(X=k)\, P(Y=j).$$

Which means that X and Y are independent Poisson random variable.

18. Consider the following joint probability density function p.d.f. of X and Y :

$$f(x,y)=\begin{cases}4xy & \text{if } 0<y<1,\ 0<x<1,\\ 0 & \text{otherwise}\end{cases}$$

(a) Find i. $P(0<X\leq 0.5, 0.25\leq Y\leq 0.5)$, ii. $P(0<Y<1)$.
i) We have

$$\begin{aligned}P(0<X\leq 0.5, 0.25\leq Y\leq 0.5) &= \int_0^{0.5}\int_{0.25}^{0.5} 4xydxdy\\ &= \int_0^{0.5} 2xdx\int_{0.25}^{0.5} 2ydy\\ &= \left[x^2\right]_0^{0.5}\times\left[y^2\right]_{0.25}^{0.5}\\ &= \frac{5^2}{10^2}\left(\frac{5^2}{10^2}-\frac{(5^2)^2}{100^2}\right)\\ &= \frac{5^4}{10^4}\left(1-\frac{5^2}{10^2}\right)=\frac{3}{64}\end{aligned}$$

ii) We have

$$\begin{aligned}P(0<Y<1) &= P(-\infty<X<+\infty, 0<Y<1)\\ &= \int_0^1\int_0^1 4xydxdy=\int_0^1 2xdx\int_0^1 2ydy\\ &= \left[x^2\right]_0^1\times\left[y^2\right]_0^1=1\end{aligned}$$

(b) Find the joint cumulative distribution function c.d.f. of X and Y , i.e., $F_{(X,Y)}(x,y) = P(X\leq x, Y\leq y)$.
By definition of the c.d.f.

$$\begin{aligned}F_{(X,Y)}(x,y) &= \int_{-\infty}^{x}\int_{-\infty}^{y} 4uvdudv\\ &= \int_{-\infty}^{x} 2udu\int_{-\infty}^{y} 2vdv\end{aligned}$$

i. if $x\leq 0$ or $y\leq 0$, then

$$F_{(X,Y)}(x,y)=0, F_{(X,Y)}(x,y)=0,$$

ii. if $0<x<1$ and $0<y<1$, then

$$\begin{aligned}F_{(X,Y)}(x,y) &= \int_0^x 2udu\int_0^y 2vdv\\ &= x^2y^2\end{aligned}$$

iii. if $0<x<1$ and $y\geq 1$, then

$$\begin{aligned}F_{(X,Y)}(x,y) &= \int_0^x 2udu\int_0^1 2vdv\\ &= x^2\end{aligned}$$

iv. if $0<y<1$ and $x\geq 1$, then

$$\begin{aligned}F_{(X,Y)}(x,y) &= \int_0^1 2udu\int_0^y 2vdv\\ &= y^2\end{aligned}$$

v. if $y \geq 1$ and $x \geq 1$, then

$$F_{(X,Y)}(x,y) = 1.$$

(c) Find i. $P(X = Y)$, ii. $P(X \leq Y)$, iii. $P(X - Y > 1/2)$

19. Consider the following joint p.d.f of X and Y :

$$f(x,y) = \begin{cases} c(x+3y) & \text{if } 0 < y < 1,\ 0 < x < 1, \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the value of c

(b) Find the marginal density function of X

(c) Find the joint c.d.f. of X and Y

(d) Find $P(0 < X < 1/2, 0 < Y < 1/2)$

(e) Find $P(X < Y)$ and $P(X + Y < 1/2)$

20. Suppose X and Y are continuous random variables with joint c.d.f. given by $F(x,y)$. For each of the following, find the answer in terms of $F(x,y)$.

(a) $P(X \leq a, Y \leq c) = F(a,c)$

(b) $P(X \leq b) = \lim_{y \longrightarrow +\infty} F(b,y)$, $P(Y \leq d) = \lim_{x \longrightarrow +\infty} F(x,d)$

(c) $P(a < X \leq b, c < Y \leq d) = F(b,d) - F(b,c) - F(a,d) + F(a,c)$

(d) $P(X > a, Y \leq d) = P(a < X < +\infty, -\infty < Y \leq d)$
$= F(+\infty,d) - F(+\infty,-\infty) - F(a,d) + F(a,-\infty) = F(+\infty,d) - F(a,d)$ (since $F(+\infty,-\infty) = F(a,-\infty) = 0$)

(e) $P(X > b, Y > d) = P(b < X < +\infty, d < Y < +\infty) =$
$= 1 - F(+\infty,d) - F(b,+\infty) + F(b,d)$

21. Suppose the joint c.d.f. of two random variables X and Y is given by

$$F(x,y) = \begin{cases} 1 - e^{-x} - e^{-y} + e^{-x-y}, & \text{for } x,y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find $P(X < 2, Y < 2)$,
By definition of the c.d.f.

$$\begin{aligned} P(X < 2, Y < 2) &= F_{(X,Y)}(2,2) = 1 - e^{-2} - e^{-2} + e^{-4} \\ &= 1 - 2e^{-2} + e^{-4} \end{aligned}$$

(b) Find $P(X < 5)$,

$$\begin{aligned} P(X < 5) &= P(X < 5, Y < +\infty) = \lim_{y \longrightarrow \infty} F(5,y) \\ &= 1 - e^{-5} \end{aligned}$$

(c) Find $P(1 < X < 3, 2 < Y < 4)$:

$$P(1 < X < 3, 2 < Y < 4) = F(3,4) - F(3,2) - F(1,4) + F(1,2)$$

(d) Find the joint p.d.f. $f_{(X,Y)}(x,y)$

$$f_{(X,Y)}(x,y) = \frac{\partial^2 F}{\partial x \partial y}(x,y) = \begin{cases} e^{-x-y}, & \text{for } x,y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

(e) Find the p.d.f $f_X(x)$ of X and the p.d.f $f_Y(y)$ of Y.

$$\begin{aligned} f_X(x) &= \int_0^\infty f_{(X,Y)}(x,y)dy = e^{-x}\int_0^\infty e^{-y}dy \\ &= e^{-x} \end{aligned}$$

and

$$\begin{aligned} f_Y(x) &= \int_0^\infty f_{(X,Y)}(x,y)dx = e^{-y}\int_0^\infty e^{-x}dx \\ &= e^{-y} \end{aligned}$$