King Saud University
College of Sciences
Department of Mathematics

First Mid Term, S2 1438/1439
M 380 – Stochastic Processes
Time: 90 minutes

Answer the following questions:

Q1: [6+3]

a) Determine the distribution function, mean and variance corresponding to the triangular density $f(x) = \begin{cases} x & \text{for } 0 \le x \le 1 \\ 2 - x & \text{for } 1 \le x \le 2 \\ 0 & \text{elsewhere} \end{cases}$

b) Find the moment generating function $M_X(t)$ of X, where $X \sim \text{Uniform}(a,b)$

Q2: [4+4]

a) Determine the mean and the median of an exponentially distributed random variable with parameter λ

b) If $X \sim \text{Bin}(p, N)$ and $N \sim \text{Poisson}(\lambda)$. What is the marginal distribution for X?

Q3: [4+4]

a) Let X and Y are jointly distributed random variables having the density function $f_{XY}(x,y) = \frac{1}{y} e^{-(x/y)-y}$ for $x,y > 0$ find $f_{XY}(x|y)$

b) Suppose that a number of miles that a car can run before its battery wears out is exponentially distributed with an average value of 10000 miles. If a person desires to take a 5000-mile trip, what is the probability that he will be able to complete his trip without having to replace the car battery?

First Ma...

Model Answer – M380

$\frac{7}{6}$

9

Q1

a) $F(x) = \int_0^x t\,dt = \frac{x^2}{2}$, $0 \leq x < 1$

$F(x) = \int_1^x (2-t)\,dt = \left[2t - \frac{t^2}{2}\right]_1^x$

$= \frac{-1}{2}(x^2 - 4x + 3)$

$= \frac{1}{2}\left[1 - (x-2)^2\right]$, $1 \leq x < 2$

$\therefore F(x) = \begin{cases} 0 & , x < 0 \\ \frac{1}{2}x^2 & , 0 \leq x < 1 \\ \frac{1}{2}\left[1 - (2-x)^2\right] & , 1 \leq x < 2 \\ 1 & , x > 2 \end{cases}$

$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$

$= \int_0^1 x(x)\,dx + \int_1^2 x(2-x)\,dx$

$= \left[\frac{x^3}{3}\right]_0^1 + \left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_1^2$

$E(X) = 1$

$Var(X) = E(X^2) - \mu^2$

$E(X^2) = \int x^2 f(x)\,dx$

$= \int_0^1 x^3\,dx + \int_1^2 x^2(2-x)\,dx$

$E(X^2) = \left[\frac{x^4}{4}\right]_0^1 + \left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_1^2$

$= 7/6$

$V(X) = \frac{1}{2} + \frac{14}{3} - 4 - 1$

$= \frac{1}{6}$

b) $M_X(t) = E(e^{Xt})$

$= \int_{-\infty}^{\infty} e^{xt} f(x)\,dx$

$= \int_a^b e^{xt} \frac{1}{b-a}\,dx$, $X \sim Uniform(a,b)$

$= \frac{1}{b-a}\left[\frac{e^{xt}}{t}\right]_a^b$

$= \frac{1}{b-a}\left(\frac{e^{bt} - e^{at}}{t}\right)$

$\therefore M_X(t) = \frac{1}{t(b-a)}(e^{bt} - e^{at})$, $t \neq 0$

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(8)

Q(2)

a) $X \sim exp(\lambda)$, $\lambda > 0$

$$f(x|\lambda) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & else \end{cases}$$

$$\mu = E(X) = \lambda \int_0^\infty x e^{-\lambda x} dx$$

let $u = \lambda x \Rightarrow x = \frac{u}{\lambda}$, $dx = \frac{du}{\lambda}$

$$\mu = \frac{\lambda}{\lambda^2} \int_0^\infty u e^{-u} du$$

$$\therefore \mu = \frac{1}{\lambda} \Gamma(2) = \frac{1}{\lambda} \text{ (mean)}$$

(4)

For $pr(X \leq a) \geq \frac{1}{2}$

$$1 - e^{-\lambda a} \geq \frac{1}{2}$$

$$-e^{-\lambda a} \geq -\frac{1}{2}$$

$$e^{-\lambda a} \leq \frac{1}{2}$$

$$\therefore a \geq \frac{\ln 2}{\lambda} \quad (1)$$

For $pr(X \geq a) \geq \frac{1}{2}$

$$e^{-\lambda a} \geq \frac{1}{2}$$

$$\therefore a \leq \frac{\ln 2}{\lambda} \quad (2)$$

(1), (2) $\Rightarrow$ Median $= \frac{\ln 2}{\lambda}$

i.e. Median < Mean for the r.v $X \sim exp(\lambda)$

b) $X \sim Bin(p, N)$, $N \sim poisson(\lambda)$

$$P_{XN}(x|n) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$P_N(n) = \frac{e^{-\lambda} \lambda^n}{n!}, n = 0, 1, 2, \ldots$$

$$pr(X = x) = \sum_{n=0}^{\infty} P_{XN}(x|n) P_N(n)$$

total prob.

Subs. (1), (2) in (3) $\Rightarrow$

(4)

$$pr(X = x) = \sum_{n=x}^{\infty} \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x} \cdot \frac{\lambda^n e^{-\lambda}}{n!}$$

$$= \frac{\lambda^x e^{-\lambda} p^x}{x!} \sum_{n=x}^{\infty} \frac{[\lambda(1-p)]^{n-x}}{(n-x)!}$$

$$= \frac{(\lambda p)^x e^{-\lambda}}{x!} \sum_{r=0}^{\infty} \frac{[\lambda(1-p)]^r}{r!}$$

where $r = n - x$

$$pr(X = x) = \frac{(\lambda p)^x e^{-\lambda} e^{\lambda(1-p)}}{x!}$$

$$pr(X = x) = \frac{(\lambda p)^x e^{-\lambda p}}{x!}, x = 0, 1, 2, \ldots$$

$\therefore X \sim poisson(\lambda p)$ with mean λp

#

3

(8)

Q3

a) $f_{XY}(x,y) = \frac{1}{y} e^{-(x/y)-y}$

for $x, y > 0$

$f_{XY}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)}$

(4)

$f_Y(y) = \int_0^{\infty} f(x,y)\,dx$

$f_Y(y) = \int_0^{\infty} \frac{1}{y} e^{-(x/y)-y}\,dx$

$f_Y(y) = \frac{e^{-y}}{y} \int_0^{\infty} e^{-x/y}\,dx$

$f_Y(y) = \frac{e^{-y}}{y} \left[\frac{e^{-x/y}}{-1/y} \right]_0^{\infty}$

$f_Y(y) = e^{-y}\,[0+1] = e^{-y}, \; y > 0$

Note: $\lim_{x\to\infty} e^{-x} = 0$

b) $X \sim \exp\left(\frac{1}{10000}\right)$

$pr(X > 5000)$

$= e^{-\frac{5000}{10000}}$

$= e^{-0.5} \simeq 0.6065$

(4)