



Answer the following questions:

Q1: [4+4]

(a) For the Markov process $\{X_t\}$, $t=0,1,2,\dots,n$ with states $i_0, i_1, i_2, \dots, i_{n-1}, i_n$

Prove that: $\Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \dots, X_n = i_n\} = p_{i_0} P_{i_0 i_1} P_{i_1 i_2} \dots P_{i_{n-1} i_n}$ where $p_{i_0} = \Pr\{X_0 = i_0\}$

(b) A Markov chain X_0, X_1, X_2, \dots has the transition probability matrix

$$\mathbf{P} = \begin{matrix} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{vmatrix} 0.2 & 0.3 & 0.5 \\ 0.4 & 0.2 & 0.4 \\ 0.5 & 0.3 & 0.2 \end{vmatrix} \end{matrix}$$

and initial distribution $p_0=0.5$, $p_1=0.2$ and $p_2=0.3$ Determine the probabilities

$$\Pr\{X_0 = 1, X_1 = 1, X_2 = 0\} \quad \text{and} \quad \Pr\{X_1 = 1, X_2 = 1, X_3 = 0\}$$

Q2: [4+4]

(a) Consider a spare parts inventory model in which either 0, 1, or 2 repair parts are demanded in any period, with $\Pr\{\xi_n = 0\} = 0.3$, $\Pr\{\xi_n = 1\} = 0.2$, $\Pr\{\xi_n = 2\} = 0.5$ and suppose $s=0$ and $S=3$. Determine the transition probability matrix for the Markov chain $\{X_n\}$, where X_n is defined to be the quantity on hand at the end of period n .

(b) For modelling weather phenomenon, let $\{X_n\}$ be a Markov chain with state space $S = \{1, 2\}$ where 1 stands for rainy and 2 stands for dry. The transition probability matrix is given by

$$\mathbf{P} = \begin{matrix} & \begin{matrix} 1 & 2 \end{matrix} \\ \begin{matrix} 1 \\ 2 \end{matrix} & \begin{vmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{vmatrix} \end{matrix}$$

Initially, assume that the probability of weather will be rainy on 1st June equals 3/8.

Find the probability for each of the following:

- (i) The weather will be dry on 2nd June.
- (ii) The weather will be dry on 3rd June.
- (iii) The weather will be rainy on 5th June.

Q3: [4+5]

(a) Suppose that the social classes of successive generations in a family follow a Markov chain with transition probability matrix given by

		Son's class		
		Lower	Middle	Upper
Father's class	Lower	0.7	0.2	0.1
	Middle	0.2	0.6	0.2
	Upper	0.1	0.4	0.5

What fraction of families are middle class in the long run?

(b) Consider the Markov chain whose transition probability matrix is given by

$$\mathbf{P} = \begin{matrix} & \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} & \left\| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0.1 & 0.6 & 0.1 & 0.2 \\ 0.2 & 0.3 & 0.4 & 0.1 \\ 0 & 0 & 0 & 1 \end{array} \right\| \end{matrix}$$

- (i) Starting in state 1, determine the probability that the Markov chain ends in state 0.
- (ii) Determine the mean time to absorption.
- (iii) Sketch, the Markov chain diagram, and determine whether it's an absorbing chain or not.



The Model Answer

Q1: [4+4]

(a)

$$\begin{aligned} & \because \Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \dots, X_n = i_n\} \\ &= \Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \dots, X_{n-1} = i_{n-1}\} \cdot \Pr\{X_n = i_n | X_0 = i_0, X_1 = i_1, X_2 = i_2, \dots, X_{n-1} = i_{n-1}\} \\ &= \Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \dots, X_{n-1} = i_{n-1}\} \cdot P_{i_{n-1}i_n} \quad \text{Definition of Markov} \end{aligned}$$

By repeating this argument $n-1$ times

$$\begin{aligned} & \therefore \Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \dots, X_n = i_n\} \\ &= p_{i_0} P_{i_0 i_1} P_{i_1 i_2} \dots P_{i_{n-2} i_{n-1}} P_{i_{n-1} i_n} \quad \text{where } p_{i_0} = \Pr\{X_0 = i_0\} \text{ is obtained from the initial distribution of the process.} \end{aligned}$$

(b)

$$\begin{aligned} \text{i) } \Pr\{X_0 = 1, X_1 = 1, X_2 = 0\} &= p_1 P_{11} P_{10}, \quad p_1 = \Pr\{X_0 = 1\} \\ &= 0.2(0.2)(0.4) \\ &= 0.016 \end{aligned}$$

$$\begin{aligned} \text{ii) } \Pr\{X_1 = 1, X_2 = 1, X_3 = 0\} &= p_1 P_{11} P_{10}, \quad p_1 = \Pr\{X_1 = 1\} \\ \Pr\{X_1 = 1\} &= \Pr(X_1 = 1 | X_0 = 0) \Pr(X_0 = 0) + \Pr(X_1 = 1 | X_0 = 1) \Pr(X_0 = 1) + \Pr(X_1 = 1 | X_0 = 2) \Pr(X_0 = 2) \\ &= P_{01} p_0 + P_{11} p_1 + P_{21} p_2 \\ &= 0.3(0.5) + 0.2(0.2) + 0.3(0.3) = 0.28 \end{aligned}$$

$$\therefore \Pr\{X_1 = 1, X_2 = 1, X_3 = 0\} = 0.28(0.2)(0.4) = 0.0224$$

Q2: [4+4]

(a)

$$\begin{array}{c} -1 \quad 0 \quad 1 \quad 2 \quad 3 \\ -1 \left\| \begin{array}{ccccc} 0 & 0 & 0.5 & 0.2 & 0.3 \\ 0 & 0 & 0.5 & 0.2 & 0.3 \\ 1 & 0.5 & 0.2 & 0.3 & 0 & 0 \\ 2 & 0 & 0.5 & 0.2 & 0.3 & 0 \\ 3 & 0 & 0 & 0.5 & 0.2 & 0.3 \end{array} \right. \end{array}$$

$$P_{ij} = \Pr(\xi_{n+1} = S - j) \quad , \quad i \leq s \quad \text{for replenishment}$$

$$P_{-1,-1} = \Pr(\xi_{n+1} = 4) = 0 \quad , \quad P_{01} = \Pr(\xi_{n+1} = 2) = 0.5$$

$$P_{ij} = \Pr(\xi_{n+1} = i - j) \quad , \quad s < i \leq S \quad \text{for non-replenishment}$$

$$P_{1,-1} = \Pr(\xi_{n+1} = 2) = 0.5 \quad , \quad P_{11} = \Pr(\xi_{n+1} = 0) = 0.3, \quad P_{21} = \Pr(\xi_{n+1} = 1) = 0.2$$

(b)

The Markov chain X_0, X_1, X_2, \dots represents the day's weather

$$\therefore \Pr(X_0 = 1) = p_2^0 = 3/8$$

$$\therefore \Pr(X_0 = 2) = p_1^0 = 5/8$$

$\Rightarrow p^0 = [3/8 \quad 5/8]$ which is the initial probability distribution

(i) The prob. of weather will be dry on 2nd June is

$$\begin{aligned} \therefore \Pr(X_1 = 2) &= p_2^1 \\ &= [3/8 \quad 5/8] \begin{bmatrix} 0.2 \\ 0.6 \end{bmatrix} \\ &= 0.45 \end{aligned}$$

(ii) The prob. of weather will be dry on 3rd June is

$$\begin{aligned} \therefore \Pr(X_2 = 2) &= \mathbf{p}_2^2 \\ \mathbf{P}^2 &= \begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{bmatrix} \begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{bmatrix} \\ &= \begin{bmatrix} 0.72 & 0.28 \\ 0.56 & 0.44 \end{bmatrix} \\ \therefore \Pr(X_2 = 2) &= [3/8 \quad 5/8] \begin{bmatrix} 0.28 \\ 0.44 \end{bmatrix} \\ &= 0.38 \end{aligned}$$

(iii) The prob. of weather will be rainy on 5th June is

$$\begin{aligned} \therefore pr(X_4 = 1) &= \mathbf{p}_1^4 \\ \mathbf{P}^4 &= \begin{bmatrix} 0.72 & 0.28 \\ 0.56 & 0.44 \end{bmatrix} \begin{bmatrix} 0.72 & 0.28 \\ 0.56 & 0.44 \end{bmatrix} \\ &= \begin{bmatrix} 0.68 & 0.32 \\ 0.65 & 0.35 \end{bmatrix} \\ \therefore pr(X_4 = 1) &= [3/8 \quad 5/8] \begin{bmatrix} 0.68 \\ 0.65 \end{bmatrix} \\ &= 0.6613 \end{aligned}$$

Q3: [4+5]

(a)

Let $\pi = (\pi_0, \pi_1, \pi_2)$ be the limiting distribution

\Rightarrow

$$\pi_0 = 0.7\pi_0 + 0.2\pi_1 + 0.1\pi_2$$

$$\pi_1 = 0.2\pi_0 + 0.6\pi_1 + 0.4\pi_2$$

$$\pi_2 = 0.1\pi_0 + 0.2\pi_1 + 0.5\pi_2$$

$$\pi_0 + \pi_1 + \pi_2 = 1$$

Solving the following equations

$$3\pi_0 - 2\pi_1 - \pi_2 = 0 \quad (1)$$

$$\pi_0 + 2\pi_1 - 5\pi_2 = 0 \quad (2)$$

$$\pi_0 + \pi_1 + \pi_2 = 1 \quad (3)$$

We get $\pi_0 = \frac{6}{17}$, $\pi_1 = \frac{7}{17}$, $\pi_2 = \frac{4}{17}$

\therefore In the long run, approximately 41.2% of families are middle class.

(b)

$$\mathbf{P} = \begin{matrix} & \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} & \left\| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0.1 & 0.6 & 0.1 & 0.2 \\ 0.2 & 0.3 & 0.4 & 0.1 \\ 0 & 0 & 0 & 1 \end{array} \right\| \end{matrix}$$

$$u_i = pr\{X_T = 0 | X_0 = i\} \quad \text{for } i=1,2,$$

$$\text{and } v_i = E[T | X_0 = i] \quad \text{for } i=1,2.$$

(i)

$$u_1 = p_{10} + p_{11}u_1 + p_{12}u_2$$

$$u_2 = p_{20} + p_{21}u_1 + p_{22}u_2$$

\Rightarrow

$$u_1 = 0.1 + 0.6u_1 + 0.1u_2$$

$$u_2 = 0.2 + 0.3u_1 + 0.4u_2$$

\Rightarrow

$$4u_1 - u_2 = 1 \quad (1)$$

$$3u_1 - 6u_2 = -2 \quad (2)$$

Solving (1) and (2), we get

$$u_1 = \frac{8}{21} \quad \text{and} \quad u_2 = \frac{11}{21}$$

Starting in state 1, the probability that the Markov chain ends in state 0 is

$$u_1 = u_{10} = \frac{8}{21}$$

$$\approx 0.38$$

(ii) Also, the mean time to absorption can be found as follows

$$v_1 = 1 + p_{11}v_1 + p_{12}v_2$$

$$v_2 = 1 + p_{21}v_1 + p_{22}v_2$$

\Rightarrow

$$v_1 = 1 + 0.6v_1 + 0.1v_2$$

$$v_2 = 1 + 0.3v_1 + 0.4v_2$$

⇒

$$4v_1 - v_2 = 10 \quad (1)$$

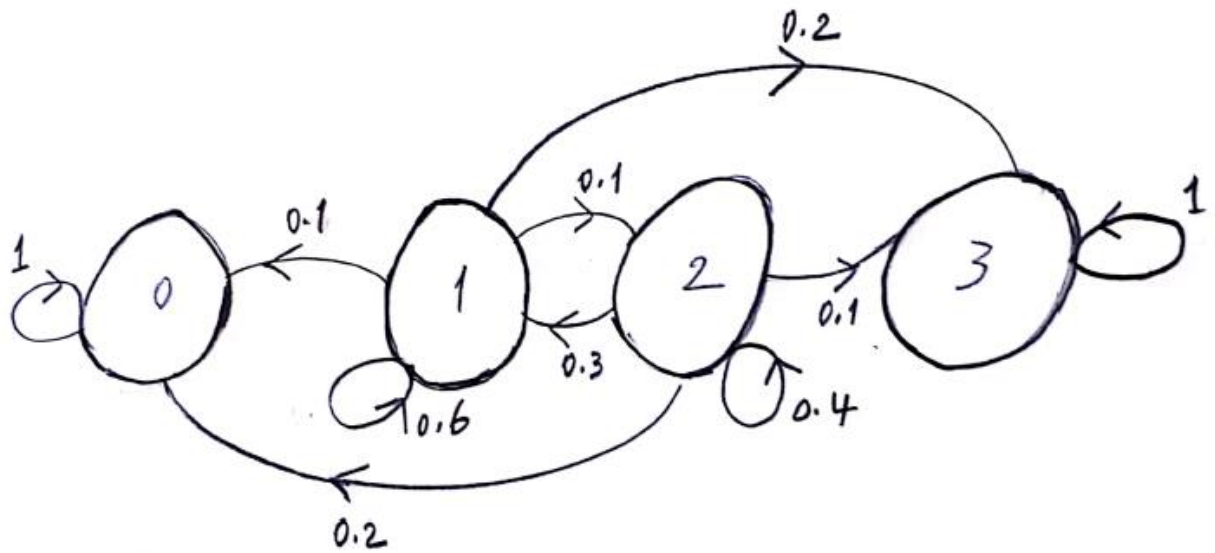
$$3v_1 - 6v_2 = -10 \quad (2)$$

Solving (1) and (2), we get

$$v_1 = v_2 = \frac{10}{3}$$

$$\therefore v_1 = v_2 = \frac{10}{3} \\ \approx 3.3$$

(iv) It's an absorbing Markov Chain.



Markov Chain Diagram
