King Saud University
College of Sciences
Department of Mathematics

Second Mid Term Exam, S2
1439/1440
M 380 – Stochastic Processes
Time: 90 minutes

Answer the following questions:

Q1: [4+4]

a) For the Markov process $\{X_t\}$, $t=0,1,2,\ldots,n$ with states $i_0, i_1, i_2, \ldots, i_{n-1}, i_n$

Prove that: $\Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_n = i_n\} = p_{i_0} P_{i_0 i_1} P_{i_1 i_2} \ldots P_{i_{n-1} i_n}$ where $p_{i_0} = \mathrm{pr}\{X_0 = i_0\}$

b) A Markov chain $X_0, X_1, X_2, \ldots$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{Vmatrix} 0.2 & 0.3 & 0.5 \\ 0.4 & 0.2 & 0.4 \\ 0.5 & 0.3 & 0.2 \end{Vmatrix} \end{array}$$

and initial distribution $p_0 = 0.5$ and $p_1 = 0.5$ Determine the probabilities

$\mathrm{pr}\{X_0 = 1, X_1 = 1, X_2 = 0\}$ and $\mathrm{pr}\{X_1 = 1, X_2 = 1, X_3 = 0\}$

Q2: [6+3]

a) Consider a spare parts inventory model in which either $0, 1,$ or 2 repair parts are demanded in any period, with $\Pr\{\xi_n = 0\} = 0.3$, $\Pr\{\xi_n = 1\} = 0.2$, $\Pr\{\xi_n = 2\} = 0.5$ and suppose $s=0$ and $S=3$. Determine the transition probability matrix for the Markov chain $\{X_n\}$, where X_n is defined to be the quantity on hand at the end of period n.

b) Let $\{X_n\}$ be a Markov chain for daily weather with two states 0, 1 (0 for dry day and 1 for rainy day) has the transition probability matrix

$$P = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{Vmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{Vmatrix} \end{array}$$

What's the probability for the weather to be dry today and rainy on the coming two days?

Q3: [8]

A Markov chain $X_0, X_1, X_2, \ldots$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} 0.5 & 0.2 & 0.3 \\ 0.5 & 0.1 & 0.4 \\ 0.3 & 0.2 & 0.5 \end{array} \right\| \end{array}$$

Every period that the process spends in state 0 incurs a cost \$4. Every period that the process spends in state 1 incurs a cost of \$7. Every period that the process spends in state 2 incurs a cost of \$5. What is the long run mean cost per period associated with this Markov chain?

Q1: [4+4]

a)

$\because \Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_n = i_n\}$

$= \Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_{n-1} = i_{n-1}\}.\Pr\{X_n = i_n | X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_{n-1} = i_{n-1}\}$

$= \Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_{n-1} = i_{n-1}\}.P_{i_{n-1}i_n}$ Definition of Markov

By repeating this argument $n-1$ times

$\therefore \Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_n = i_n\}$

$= p_{i_0} P_{i_0 i_1} P_{i_1 i_2} \ldots P_{i_{n-2} i_{n-1}} P_{i_{n-1} i_n}$ where $p_{i_0} = \Pr\{X_0 = i_0\}$ is obtained from the initial distribution of the process.

b)

i) $\mathrm{pr}\{X_0 = 1, X_1 = 1, X_2 = 0\} = p_1 P_{11} P_{10}$, $p_1 = \mathrm{pr}\{X_0 = 1\}$

$= 0.5(0.2)(0.4)$

$= 0.04$

ii) $\mathrm{pr}\{X_1 = 1, X_2 = 1, X_3 = 0\} = p_1 P_{11} P_{10}$, $p_1 = \mathrm{pr}\{X_1 = 1\}$

$\mathrm{pr}\{X_1 = 1\} = \Pr(X_1 = 1 | X_0 = 0)\Pr(X_0 = 0) + \Pr(X_1 = 1 | X_0 = 1)\Pr(X_0 = 1) + \Pr(X_1 = 1 | X_0 = 2)\Pr(X_0 = 2)$

$= P_{01} p_0 + P_{11} p_1 + P_{21} p_2$

$= 0.3(0.5) + 0.2(0.5) = 0.25$

$\therefore \mathrm{pr}\{X_1 = 1, X_2 = 1, X_3 = 0\} = 0.25(0.2)(0.4) = 0.02$

Q2: [6+3]

a)

	-1	0	1	2	3
-1	0	0	0.5	0.2	0.3
0	0	0	0,5	0.2	0.3
1	0.5	0.2	0.3	0	0
2	0	0.5	0.2	0.3	0
3	0	0	0,5	0.2	0.3

$P_{ij} = \Pr(\xi_{n+1} = S - j)$, $i \le s$ for replenishment

$P_{-1,-1} = \Pr(\xi_{n+1} = 4) = 0$, $P_{01} = \Pr(\xi_{n+1} = 2) = 0.5$

$P_{ij} = \Pr(\xi_{n+1} = i - j)$, $s < i \le S$ for non-replenishment

$P_{1,-1} = \Pr(\xi_{n+1} = 2) = 0.5$, $P_{11} = \Pr(\xi_{n+1} = 0) = 0.3$, $P_{21} = \Pr(\xi_{n+1} = 1) = 0.2$

b)

$$\Pr(X_2=1 | X_0=0) = P_{01}^2$$

$$P^2 = \begin{Vmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{Vmatrix} \begin{Vmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{Vmatrix}$$

$$= \begin{Vmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{Vmatrix}$$

$$\therefore \Pr(X_2=1 | X_0=0) = 0.39$$

Q3: [8]

$$\pi_j = \sum_{k=0}^{2} \pi_k P_{kj}$$

at $j=0$

$$\Rightarrow \pi_0 = 0.5\pi_0 + 0.5\pi_1 + 0.3\pi_2$$

$$\therefore 5\pi_0 - 5\pi_1 - 3\pi_2 = 0 \qquad (1)$$

at $j=1$

$$\Rightarrow \pi_1 = 0.2\pi_0 + 0.1\pi_1 + 0.2\pi_2$$

$$\therefore 2\pi_0 - 9\pi_1 + 2\pi_2 = 0 \qquad (2)$$

and $\because \pi_0 + \pi_1 + \pi_2 = 1 \qquad (3)$

$\therefore$ By solving equations (1), (2) and (3)

We get $\pi_0 = 0.4205, \ \pi_1 = 0.1818, \ \pi_2 = 0.3977$

The long run mean cost per unit period is

$$C = \sum_{j=0}^{2} \pi_j c_j$$

$$= \pi_0 c_0 + \pi_1 c_1 + \pi_2 c_2$$

$$= \$\ 4.9431$$