King Saud University
College of Sciences
Department of Mathematics

Mid-Term Exam, S2 1439/1440
M 507 –Operation Research
Time: 90 minutes

Answer the following questions:

Q1: [4+4]

a) Given the following joint distribution. Calculate $E(X)$, $E(Y)$, $Var(X)$, $Var(Y)$, $Cov(X,Y)$, $\rho(X,Y)$, and verify $E(X)$ using the law of total Expectation.

Y \ X	0	1
0	0.1	0.3
1	0.4	0.2

b) The lifetime, in years, of a certain class of light bulbs has an exponential distribution with parameter $\lambda = 2$. What is the probability that a bulb selected at random from this class will last more than 1.5 years? What is the probability that a bulb selected at random will last exactly 1.5 years?

Q2: [4+4+4]

a) Prove that each of the following:

i) $R(t) = e^{-\int_0^t \lambda(u)\, du}$ ii) MTTF= $\int_0^\infty R(t)\, dt$

b) Prove that: the average failure rate over the interval (t_1, t_2) is given by $\bar{\lambda} = \frac{\Lambda(t_2) - \Lambda(t_1)}{t_2 - t_1}$, and then derive the average failure rate for the two parameter Weibull distribution.

c) The life of a product follows a Weibull distribution with a shape parameter of 1.5 and a scale parameter of 1000 hours. Compute the instantaneous failure rate at its characteristic value and the average failure rate over the time interval $(1000, 2000)$ in hours.

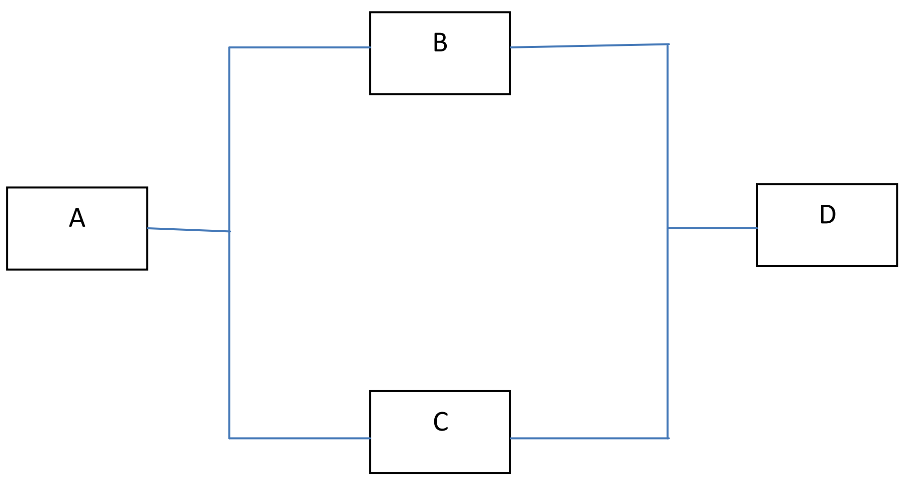
Q3: [4+2+4]

In the reliability diagram below, the reliability of each component is constant and independent. Assuming that each has the same reliability R, compute the system reliability as a function of R using the following methods:

a. Decomposition using B as the keystone element.

b. The reduction method.

c. Compute the importance of each component if $R_A = 0.98$, $R_B = 0.95$, $R_C = 0.9$ and $R_D = 0.98$



The Model Answer

Q1: [4+4]

a)

Y \ X	0	1	$P_Y(y)$
0	0.1	0.3	0.4
1	0.4	0.2	0.6
$P_X(x)$	0.5	0.5	Sum=1

$E(X)=0.5,\ E(X^2)=0.5,\ Var(X)=0.25$

$E(Y)=0.6,\ E(Y^2)=0.6,\ Var(Y)=0.24$

$E(XY)=0.2,\ Cov(X,Y)=-0.10,\ \rho(X,Y)=-0.4$

$$P(X|Y=y)=\frac{P_{X,Y}(x,y)}{P_Y(y)}$$

$$P(X=0|Y=0)=\frac{0.1}{0.4}=\frac{1}{4},\quad P(X=1|Y=0)=\frac{0.3}{0.4}=\frac{3}{4}$$

$$P(X=0|Y=1)=\frac{0.4}{0.6}=\frac{2}{3},\quad P(X=1|Y=1)=\frac{0.2}{0.6}=\frac{1}{3}$$

$X|Y$	0	1	$E[X|Y]$
y=0	1/4	3/4	3/4
y=1	2/3	1/3	1/3

$\mathrm{E(X)}=\sum_{y}\mathrm{E(X|Y=y)P_Y(y)}$

$\mathrm{E(X)}=\frac{3}{4}\mathrm{P_Y(0)}+\frac{1}{3}\mathrm{P_Y(1)}$

$\mathrm{E(X)}=\frac{3}{4}(0.4)+\frac{1}{3}(0.6)=0.5$

b） $X\sim \exp(2)$

i） $\Pr(T>1.5)=e^{-3}=0.0498$

ii） $\Pr(T=1.5)=0$

Q2: [4+4+4]

a）

i） To prove that $R(t)=e^{-\int_0^t \lambda(u)\,du}$

The hazard function or failure rate is given by $\lambda(t)=\frac{f_T(T)}{R(t)}$

$$\Rightarrow \begin{array}{l}\lambda(t)=\frac{d}{dt}F_T(t).\frac{1}{R(t)}\\ \lambda(t)=-\frac{d}{dt}R(t).\frac{1}{R(t)}\end{array}$$

$$\Rightarrow \int_0^t \frac{dR(u)}{R(u)}=-\int_0^t \lambda(u)du$$

$$\therefore\ [\ln R(u)]_0^t=-\int_0^t \lambda(u)du$$

$$\therefore\ \ln R(t)-\ln R(0)=-\int_0^t \lambda(u)du\ ,\ \because\ \ln R(0)=\ln(1)=0$$

$$\therefore \ \ln R(t) = -\int_0^t \lambda(u)du$$

$$\therefore \ R(t) = e^{-\int_0^t \lambda(u)du}$$

$$\therefore \ R(t) = \exp[-\int_0^t \lambda(u)du]$$

ii） To prove that MTTF= $\int_0^\infty t \, f_T(t)dt$

MTTF=-$\int_0^\infty t \ \frac{dR(t)}{dt} dt$

MTTF=-$\int_0^\infty t \ dR(t)$

By using Integration by parts, we deduce that

MTTF=-$[tR(t)]_0^\infty + \int_0^\infty R(t)dt$

at $t = 0 \Rightarrow tR(t) = 0 \ R(0) = 0(1) = 0$

$$\lim_{t\to\infty} tR(t) = \lim_{t\to\infty} \frac{t}{1/R(t)} \quad \to \frac{\infty}{\infty}$$

By using L'Hopital Rule we get

$$\lim_{t\to\infty} tR(t) = \lim_{t\to\infty} \frac{1}{-R'(t)/R^2(t)}$$

$$= \lim_{t\to\infty} \frac{R(t)}{-R'(t)/R(t)}$$

$$= \lim_{t\to\infty} \frac{R(t)}{\lambda(t)} = 0$$

Where R(t)=0 and λ(t) $\neq 0$ as $t \to \infty$

$\therefore$ MTTF$=\int_0^\infty R(t)dt$

b）$\because\ R(t)=e^{-\int_0^t \lambda(u)du}$

$\therefore\ R(t)=e^{-\Lambda(t)},\ \Lambda(t)=\int_0^t \lambda(u)du$

The average failure rate over the interval (t_1,t_2) is given by

$$\bar{\lambda}=\frac{\int_{t_1}^{t_2}\lambda(u)du}{t_2-t_1}$$

$$\therefore\ \bar{\lambda}=\frac{\int_0^{t_2}\lambda(u)du-\int_0^{t_1}\lambda(u)du}{t_2-t_1}$$

$$\therefore\ \bar{\lambda}=\frac{\Lambda(t_2)-\Lambda(t_1)}{t_2-t_1} \qquad (1)$$

For 2p Weibull

$\therefore\ R(t)=e^{-\Lambda(t)},\ \Lambda(t)=\int_0^t \lambda(u)du$

$R(t)=\exp[-(\frac{t}{\eta})^\beta],\ R(t)=e^{-\Lambda(t)}$

$\therefore\ \Lambda(t)=(\frac{t}{\eta})^\beta \qquad (2)$

$$\therefore\ (1),(2)\Rightarrow\ \bar{\lambda}=\frac{t_2^{\beta}-t_1^{\beta}}{\eta^{\beta}(t_2-t_1)}$$

c）

$\because\ \lambda(t)=\frac{\beta}{\eta^\beta}t^{\beta-1}$

$\therefore\ \lambda(1000)=1.5\times10^{-3}$

The average failure rate is

$$\bar{\lambda} = \frac{t_2^{\beta} - t_1^{\beta}}{\eta^{\beta}(t_2 - t_1)}$$

$$\bar{\lambda} = \frac{(2000)^{1.5} - (1000)^{1.5}}{(1000)^{1.5}(1000)} \simeq 1.8\times 10^{-3}$$

Q3: [4+2+4]

In the reliability diagram shown in Fig. 1, the reliability of each component is constant and independent. Assuming that each has the same reliability R, compute the system reliability as a function of R using the following methods:

a) Decomposition using B as the keystone element.

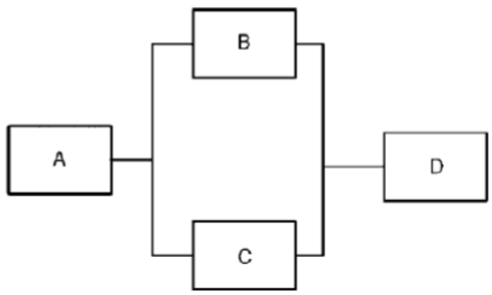


Fig. 1: Reliability diagram

Using B as the keystone element, we have two cases i.e., the case when B functions and the case when it does not.

For the case when B functions, the system reduced to Fig 2.

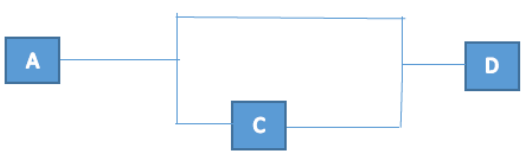


Fig. 2: The case when B functions

Thus the reliability of the system depends only on the reliability of component A and D. Note that $R_A = R_B = R_C = R_D = R$

Therefore,

$$R^{+} = R_A R_D = R^2$$

For the case when B fails, the system block is as shown in Fig. 3, which is a series system.

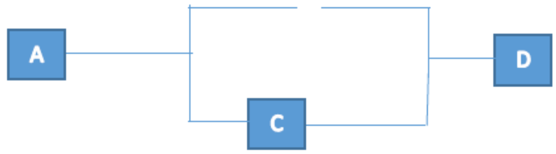


Fig. 3: The case when B fails to work

Thus the reliability of the system depends on A, C, and D, therefore we have:

$$R^{-} = R_A R_C R_D = R^3$$

Thus the reliability of the system using the two decompositions is given as:

$$R_{system} = R_B R^{+} + (1 - R_B) R^{-}$$
$$R_{system} = R(R^2) + (1 - R) R^3$$
$$R_{system} = 2R^3 - R^4$$

b) Using the reduction method

With this method, it can be seen that components B and C are in parallel and jointly in series with A and D. therefore the reduced system is given in Fig. 4.

Fig. 4: Reduced system

For parallel components B and C, we have

$$R_{B\|C} = 1 - \prod_{i=1}^{2} (1 - R_i)$$
$$R_{B\|C} = R_B + R_C - R_B R_C$$
$$R_{B\|C} = 2R - R^2$$

The reliability of the system is thus given as:

$$R_{system} = R_A R_{B\|C} R_D$$
$$R_{system} = R(2R - R^2) R$$
$$R_{system} = 2R^3 - R^4$$

Recall that the reliability of the system is given as:

$$R_{system} = R_A R_D (R_B + R_C - R_B R_C)$$

The importance of each component is computed by taking the partial derivative with respect to each of the component.

Thus the importance of component A is given as:

$$\frac{\delta R_{system}}{\delta R_A} = \frac{\delta(R_A R_D (R_B + R_C - R_B R_C))}{\delta R_A}$$

$$I_A = R_D (R_B + R_C - R_B R_C)$$

$$\Rightarrow \ I_A = 0.98(0.95 + 0.9 - 0.95 \times 0.9)$$

$$= 0.9751$$

The importance of component B is given as:

$$\frac{\delta R_{system}}{\delta R_B} = \frac{\delta(R_A R_D (R_B + R_C - R_B R_C))}{\delta R_B}$$

$$I_B = R_A R_D - R_A R_D R_C$$

$$\Rightarrow \ I_B = (0.98)^2 - (0.98)^2 (0.9)$$

$$= 0.09604$$

The importance of component C is given as:

$$\frac{\delta R_{system}}{\delta R_C} = \frac{\delta(R_A R_D (R_B + R_C - R_B R_C))}{\delta R_C}$$

$$I_C = R_A R_D - R_A R_B R_D$$

The importance of component D is given as:

$$\frac{\delta R_{system}}{\delta R_D} = \frac{\delta(R_A R_D (R_B + R_C - R_B R_C))}{\delta R_D}$$

$$I_D = R_A (R_B + R_C - R_B R_C)$$

$$\Rightarrow \ I_D = 0.98(0.95 + 0.9 - 0.95 \times 0,9)$$

$$= 0.9751$$