

Chapter 3

Determinants

- 3.1 The Determinant of a Matrix
- 3.2 Evaluation of a Determinant using Elementary Operations
- 3.3 Properties of Determinants
- 3.4 Introduction to Eigenvalues
- 3.5 Application of Determinants

3.1 The Determinant of a Matrix

- the determinant of a 2×2 matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\Rightarrow \det(A) = |A| = a_{11}a_{22} - a_{21}a_{12}$$

- Note:

$$\left| \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \right| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

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- **Ex. 1:** (The determinant of a matrix of order 2)

$$\begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} = 2(2) - 1(-3) = 4 + 3 = 7$$

$$\begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = 2(2) - 4(1) = 4 - 4 = 0$$

$$\begin{vmatrix} 0 & 3 \\ 2 & 4 \end{vmatrix} = 0(4) - 2(3) = 0 - 6 = -6$$

- **Note:** The determinant of a matrix can be positive, zero, or negative.

- Minor of the entry a_{ij} :

The determinant of the matrix determined by deleting the i th row and j th column of A

$$M_{ij} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1(j-1)} & a_{1(j+1)} & \cdots & a_{1n} \\ \vdots & & & \vdots & \vdots & & \\ a_{(i-1)1} & & \cdots & a_{(i-1)(j-1)} & a_{(i-1)(j+1)} & \cdots & a_{(i-1)n} \\ a_{(i+1)1} & & \cdots & a_{(i+1)(j-1)} & a_{(i+1)(j+1)} & \cdots & a_{(i+1)n} \\ \vdots & & & \vdots & \vdots & & \vdots \\ a_{n1} & & \cdots & a_{n(j-1)} & a_{n(j+1)} & \cdots & a_{nn} \end{vmatrix}$$

- Cofactor of a_{ij} :

$$C_{ij} = (-1)^{i+j} M_{ij}$$

■ **Ex:**

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\Rightarrow M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

$$M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

$$\Rightarrow C_{21} = (-1)^{2+1} M_{21} = -M_{21}$$

$$C_{22} = (-1)^{2+2} M_{22} = M_{22}$$

- **Notes:** Sign pattern for cofactors

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

3×3 matrix

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

4×4 matrix

$$\begin{bmatrix} + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

$n \times n$ matrix

- **Notes:**

Odd positions (where $i+j$ is odd) have negative signs, and even positions (where $i+j$ is even) have positive signs.

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- **Ex 2:** Find all the minors and cofactors of A .

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}$$

Sol: (1) All the minors of A .

$$\Rightarrow M_{11} = \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -1, \quad M_{12} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} = -5, \quad M_{13} = \begin{vmatrix} 3 & -1 \\ 4 & 0 \end{vmatrix} = 4$$

$$M_{21} = \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = 2, \quad M_{22} = \begin{vmatrix} 0 & 1 \\ 4 & 1 \end{vmatrix} = -4, \quad M_{23} = \begin{vmatrix} 0 & 2 \\ 4 & 1 \end{vmatrix} = -8$$

$$M_{31} = \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} = 5, \quad M_{32} = \begin{vmatrix} 0 & 1 \\ 3 & 2 \end{vmatrix} = -3, \quad M_{33} = \begin{vmatrix} 0 & 2 \\ 3 & -1 \end{vmatrix} = -6$$

Sol: (2) All the cofactors of A.

$$\because C_{ij} = (-1)^{i+j} M_{ij}$$

$$\Rightarrow C_{11} = + \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -1, \quad C_{12} = - \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} = 5, \quad C_{13} = + \begin{vmatrix} 3 & -1 \\ 4 & 0 \end{vmatrix} = 4$$

$$C_{21} = - \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = -2, \quad C_{22} = + \begin{vmatrix} 0 & 1 \\ 4 & 1 \end{vmatrix} = -4, \quad C_{23} = - \begin{vmatrix} 0 & 2 \\ 4 & 1 \end{vmatrix} = 8$$

$$C_{31} = + \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} = 5, \quad C_{32} = - \begin{vmatrix} 0 & 1 \\ 3 & 2 \end{vmatrix} = 3, \quad C_{33} = + \begin{vmatrix} 0 & 2 \\ 3 & -1 \end{vmatrix} = -6$$

- **Thm 3.1: (Expansion by cofactors)**

Let A is a square matrix of order n .

Then the determinant of A is given by

$$(a) \quad \det(A) = |A| = \sum_{j=1}^n a_{ij} C_{ij} = a_{i1} C_{i1} + a_{i2} C_{i2} + \cdots + a_{in} C_{in}$$

(Cofactor expansion along the i -th row, $i=1, 2, \dots, n$)

or

$$(b) \quad \det(A) = |A| = \sum_{i=1}^n a_{ij} C_{ij} = a_{1j} C_{1j} + a_{2j} C_{2j} + \cdots + a_{nj} C_{nj}$$

(Cofactor expansion along the j -th row, $j=1, 2, \dots, n$)

- Ex: The determinant of a matrix of order 3

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{aligned} \Rightarrow \det(A) &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} \\ &= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} \\ &= a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} \\ &= a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32} \\ &= a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33} \end{aligned}$$

■ Ex 3: The determinant of a matrix of order 3

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix} \quad \begin{array}{l} \text{Ex2} \\ \Rightarrow C_{11} = -1, C_{12} = 5, C_{13} = 4 \\ C_{21} = -2, C_{22} = -4, C_{23} = 8 \\ C_{31} = 5, C_{32} = 3, C_{33} = -6 \end{array}$$

Sol:

$$\begin{aligned} \Rightarrow \det(A) &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = (0)(-1) + (2)(5) + (1)(4) = 14 \\ &= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} = (3)(-2) + (-1)(-4) + (2)(8) = 14 \\ &= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} = (4)(5) + (0)(3) + (1)(-6) = 14 \\ &= a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} = (0)(-1) + (3)(-2) + (4)(5) = 14 \\ &= a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32} = (2)(5) + (-1)(-4) + (0)(3) = 14 \\ &= a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33} = (1)(4) + (2)(8) + (1)(-6) = 14 \end{aligned}$$

- Ex 5: (The determinant of a matrix of order 3)

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & -4 & 1 \end{bmatrix} \Rightarrow \det(A) = ?$$

Sol:

$$C_{11} = (-1)^{1+1} \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -1 \quad C_{12} = (-1)^{1+2} \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} = (-1)(-5) = 5$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} 3 & -1 \\ 4 & 0 \end{vmatrix} = 4$$

$$\begin{aligned} \Rightarrow \det(A) &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= (0)(-1) + (2)(5) + (1)(4) \\ &= 14 \end{aligned}$$

- **Notes:**

The row (or column) containing the most zeros is the best choice for expansion by cofactors .

- **Ex 4: (The determinant of a matrix of order 4)**

$$A = \begin{bmatrix} 1 & -2 & 3 & 0 \\ -1 & 1 & 0 & 2 \\ 0 & 2 & 0 & 3 \\ 3 & 4 & 0 & -2 \end{bmatrix} \Rightarrow \det(A) = ?$$

Sol:

$$\det(A) = (3)(C_{13}) + (0)(C_{23}) + (0)(C_{33}) + (0)(C_{43})$$

$$= 3C_{13}$$

$$= 3(-1)^{1+3} \begin{vmatrix} -1 & 1 & 2 \\ 0 & 2 & 3 \\ 3 & 4 & -2 \end{vmatrix}$$

$$= 3 \left[(0)(-1)^{2+1} \begin{vmatrix} 1 & 2 \\ 4 & -2 \end{vmatrix} + (2)(-1)^{2+2} \begin{vmatrix} -1 & 2 \\ 3 & -2 \end{vmatrix} + (3)(-1)^{2+3} \begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} \right]$$

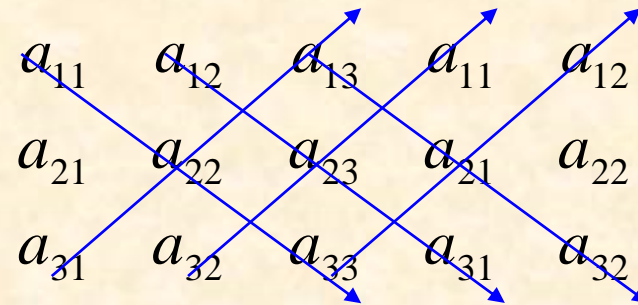
$$= 3[0 + (2)(1)(-4) + (3)(-1)(-7)]$$

$$= (3)(13)$$

$$= 39$$

- The determinant of a matrix of order 3:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$



Subtract these three products.

Add these three products.

$$\begin{aligned} \Rightarrow \det(A) = |A| &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} \\ &\quad - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12} \end{aligned}$$

■ Ex 5:

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & -4 & 1 \end{bmatrix} \begin{matrix} -4 & 0 & 6 \\ 0 & 2 \\ 3 & -1 \\ 4 & -4 \\ 0 & 16 & -12 \end{matrix}$$

$$\Rightarrow \det(A) = |A| = 0 + 16 - 12 - (-4) - 0 - 6 = 2$$

- **Upper triangular matrix:**

All the entries below the main diagonal are zeros.

- **Lower triangular matrix:**

All the entries above the main diagonal are zeros.

- **Diagonal matrix:**

All the entries above and below the main diagonal are zeros.

- **Note:**

A matrix that is both upper and lower triangular is called diagonal.

- Ex:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

upper triangular

$$\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

lower triangular

$$\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

diagonal

- **Thm 3.2: (Determinant of a Triangular Matrix)**

If A is an $n \times n$ triangular matrix (upper triangular, lower triangular, or diagonal), then its determinant is the product of the entries on the main diagonal. That is

$$\det(A) = |A| = a_{11}a_{22}a_{33} \cdots a_{nn}$$

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- **Ex 6:** Find the determinants of the following triangular matrices.

$$(a) \quad A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ -5 & 6 & 1 & 0 \\ 1 & 5 & 3 & 3 \end{bmatrix}$$

$$(b) \quad B = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

Sol:

$$(a) \quad |A| = (2)(-2)(1)(3) = -12$$

$$(b) \quad |B| = (-1)(3)(2)(4)(-2) = 48$$

Keywords in Section 3.1:

- determinant :
- minor :
- cofactor :
- expansion by cofactors :
- upper triangular matrix:
- lower triangular matrix:
- diagonal matrix:

3.2 Evaluation of a determinant using elementary operations

- **Thm 3.3: (Elementary row operations and determinants)**

Let A and B be square matrices.

$$(a) \quad B = r_{ij}(A) \quad \Rightarrow \quad \det(B) = -\det(A) \quad (\text{i.e. } |r_{ij}(A)| = -|A|)$$

$$(b) \quad B = r_i^{(k)}(A) \quad \Rightarrow \quad \det(B) = k \det(A) \quad (\text{i.e. } |r_i^{(k)}(A)| = k|A|)$$

$$(c) \quad B = r_{ij}^{(k)}(A) \quad \Rightarrow \quad \det(B) = \det(A) \quad (\text{i.e. } |r_{ij}^{(k)}(A)| = |A|)$$

▪ **Ex:**

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 1 & 2 & 1 \end{bmatrix} \quad \det(A) = -2$$

$$A_1 = \begin{bmatrix} 4 & 8 & 12 \\ 0 & 1 & 4 \\ 1 & 2 & 1 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} \quad A_3 = \begin{bmatrix} 1 & 2 & 3 \\ -2 & -3 & -2 \\ 1 & 2 & 1 \end{bmatrix}$$

$$A_1 = r_1^{(4)}(A) \Rightarrow \det(A_1) = \det(r_1^{(4)}(A)) = 4 \det(A) = (4)(-2) = -8$$

$$A_2 = r_{12}(A) \Rightarrow \det(A_2) = \det(r_{12}(A)) = -\det(A) = -(-2) = 2$$

$$A_3 = r_{12}^{(-2)}(A) \Rightarrow \det(A_3) = \det(r_{12}^{(-2)}(A)) = \det(A) = -2$$

- Notes:

$$\det(r_{ij}(A)) = -\det(A) \quad \Rightarrow \quad \det(A) = -\det(r_{ij}(A))$$

$$\det(r_i^{(k)}(A)) = k \det(A) \quad \Rightarrow \quad \det(A) = \frac{1}{k} \det(r_i^{(k)}(A))$$

$$\det(r_{ij}^{(k)}(A)) = \det(A) \quad \Rightarrow \quad \det(A) = \det(r_{ij}^{(k)}(A))$$

Note:

A row-echelon form of a square matrix is always upper triangular.

- Ex 2: (Evaluation a determinant using elementary row operations)

$$A = \begin{bmatrix} 2 & -3 & 10 \\ 1 & 2 & -2 \\ 0 & 1 & -3 \end{bmatrix} \Rightarrow \det(A) = ?$$

Sol:

$$\det(A) = \begin{vmatrix} 2 & -3 & 10 \\ 1 & 2 & -2 \\ 0 & 1 & -3 \end{vmatrix} \stackrel{r_{12}}{=} - \begin{vmatrix} 1 & 2 & -2 \\ 2 & -3 & 10 \\ 0 & 1 & -3 \end{vmatrix}$$

$$r_{12}^{(-2)} = - \begin{vmatrix} 1 & 2 & -2 \\ 0 & -7 & 14 \\ 0 & 1 & -3 \end{vmatrix} \stackrel{r_2^{(-\frac{1}{7})}}{=} (-1) \left(\frac{1}{-\frac{1}{7}} \right) \begin{vmatrix} 1 & 2 & -2 \\ 0 & 1 & -2 \\ 0 & 1 & -3 \end{vmatrix}$$

$$r_{23}^{(-1)} = 7 \begin{vmatrix} 1 & 2 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & -1 \end{vmatrix} = (7)(1)(1)(-1) = -7$$

■ Notes:

$$|EA| = |E||A|$$

$$(1) \quad E = R_{ij} \quad \Rightarrow |E| = |R_{ij}| = -1$$

$$\Rightarrow |EA| = |r_{ij}(A)| = -|A| = |R_{ij}||A| = |E||A|$$

$$(2) \quad E = R_i^{(k)} \quad \Rightarrow |E| = |R_i^{(k)}| = k$$

$$\Rightarrow |EA| = |r_i^{(k)}(A)| = k|A| = |R_i^{(k)}||A| = |E||A|$$

$$(3) \quad E = R_{ij}^{(k)} \quad \Rightarrow |E| = |R_{ij}^{(k)}| = 1$$

$$\Rightarrow |EA| = |r_{ij}^{(k)}(A)| = 1|A| = |R_{ij}^{(k)}||A| = |E||A|$$

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- Determinants and elementary column operations
 - Thm: (Elementary column operations and determinants)

Let A and B be square matrices.

$$(a) \quad B = c_{ij}(A) \quad \Rightarrow \quad \det(B) = -\det(A) \quad (\text{i.e. } |c_{ij}(A)| = -|A|)$$

$$(b) \quad B = c_i^{(k)}(A) \quad \Rightarrow \quad \det(B) = k \det(A) \quad (\text{i.e. } |c_i^{(k)}(A)| = k|A|)$$

$$(c) \quad B = c_{ij}^{(k)}(A) \quad \Rightarrow \quad \det(B) = \det(A) \quad (\text{i.e. } |c_{ij}^{(k)}(A)| = |A|)$$

▪ Ex:

$$A = \begin{bmatrix} 2 & 1 & -3 \\ 4 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix} \quad \det(A) = -8$$

$$A_1 = \begin{bmatrix} 1 & 1 & -3 \\ 2 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix} \quad A_2 = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 4 & 1 \\ 0 & 0 & 2 \end{bmatrix} \quad A_3 = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$A_1 = c_1^{(\frac{1}{2})}(A) \Rightarrow \det(A_1) = \det(c_1^{(\frac{1}{2})}(A)) = \frac{1}{2} \det(A) = \left(\frac{1}{2}\right)(-8) = -4$$

$$A_2 = c_{12}(A) \Rightarrow \det(A_2) = \det(c_{12}(A)) = -\det(A) = -(-8) = 8$$

$$A_3 = c_{23}^{(3)}(A) \Rightarrow \det(A_3) = \det(c_{23}^{(3)}(A)) = \det(A) = -8$$

- **Thm 3.4: (Conditions that yield a zero determinant)**

If A is a square matrix and any of the following conditions is true, then $\det(A) = 0$.

(a) An entire row (or an entire column) consists of zeros.

(b) Two rows (or two columns) are equal.

(c) One row (or column) is a multiple of another row (or column).

■ Ex:

$$\begin{array}{ccc|c} 1 & 2 & 3 & \\ \hline 0 & 0 & 0 & = 0 \\ \hline 4 & 5 & 6 & \end{array}$$

$$\begin{array}{ccc|c} 1 & 4 & 0 & \\ \hline 2 & 5 & 0 & = 0 \\ \hline 3 & 6 & 0 & \end{array}$$

$$\begin{array}{ccc|c} 1 & 1 & 1 & \\ \hline 2 & 2 & 2 & = 0 \\ \hline 4 & 5 & 6 & \end{array}$$

$$\begin{array}{ccc|c} 1 & 4 & 2 & \\ \hline 1 & 5 & 2 & = 0 \\ \hline 1 & 6 & 2 & \end{array}$$

$$\begin{array}{ccc|c} 1 & 2 & 3 & \\ \hline 4 & 5 & 6 & = 0 \\ \hline -2 & -4 & -6 & \end{array}$$

$$\begin{array}{ccc|c} 1 & 8 & 4 & \\ \hline 2 & 10 & 5 & = 0 \\ \hline 3 & 12 & 6 & \end{array}$$

■ Note:

Order n	Cofactor Expansion		Row Reduction	
	Additions	Multiplications	Additions	Multiplications
3	5	9	5	10
5	119	205	30	45
10	3,628,799	6,235,300	285	339

■ Ex 5: (Evaluating a determinant)

$$A = \begin{bmatrix} -3 & 5 & 2 \\ 2 & -4 & -1 \\ -3 & 0 & 6 \end{bmatrix}$$

Sol:

$$\det(A) = \begin{vmatrix} -3 & 5 & 2 \\ 2 & -4 & -1 \\ -3 & 0 & 6 \end{vmatrix} \stackrel{C_{13}^{(2)}}{=} \begin{vmatrix} -3 & 5 & -4 \\ 2 & -4 & 3 \\ -3 & 0 & 0 \end{vmatrix}$$

$$= (-3)(-1)^{3+1} \begin{vmatrix} 5 & -4 \\ -4 & 3 \end{vmatrix} = (-3)(-1) = 3$$

$$\det(A) = \begin{vmatrix} -3 & 5 & 2 \\ 2 & -4 & -1 \\ -3 & 0 & 6 \end{vmatrix} \stackrel{r_{12}^{(\frac{4}{5})}}{=} \begin{vmatrix} -3 & 5 & 2 \\ \frac{-2}{5} & 0 & \frac{3}{5} \\ -3 & 0 & 6 \end{vmatrix}$$

$$= (5)(-1)^{1+2} \begin{vmatrix} \frac{-2}{5} & \frac{3}{5} \\ -3 & 6 \end{vmatrix} = (-5)(-\frac{3}{5}) = 3$$

■ Ex 6: (Evaluating a determinant)

$$A = \begin{bmatrix} 2 & 0 & 1 & 3 & -2 \\ -2 & 1 & 3 & 2 & -1 \\ 1 & 0 & -1 & 2 & 3 \\ 3 & -1 & 2 & 4 & -3 \\ 1 & 1 & 3 & 2 & 0 \end{bmatrix}$$

Sol:

$$\det(A) = \begin{vmatrix} 2 & 0 & 1 & 3 & -2 \\ -2 & 1 & 3 & 2 & -1 \\ 1 & 0 & -1 & 2 & 3 \\ 3 & -1 & 2 & 4 & -3 \\ 1 & 1 & 3 & 2 & 0 \end{vmatrix} \stackrel{\substack{r_{24}^{(1)} \\ r_{25}^{(-1)}}}{=} \begin{vmatrix} 2 & 0 & 1 & 3 & -2 \\ -2 & 1 & 3 & 2 & -1 \\ 1 & 0 & -1 & 2 & 3 \\ 1 & 0 & 5 & 6 & -4 \\ 3 & 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= (1)(-1)^{2+2} \begin{vmatrix} 2 & 1 & 3 & -2 \\ 1 & -1 & 2 & 3 \\ 1 & 5 & 6 & -4 \\ 3 & 0 & 0 & 1 \end{vmatrix}$$

$$\begin{aligned}
& c_{41}^{(-3)} \begin{vmatrix} 8 & 1 & 3 & -2 \\ -8 & -1 & 2 & 3 \\ 13 & 5 & 6 & -4 \\ 0 & 0 & 0 & 1 \end{vmatrix} = (1)(-1)^{4+4} \begin{vmatrix} 8 & 1 & 3 \\ -8 & -1 & 2 \\ 13 & 5 & 6 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 5 \\ -8 & -1 & 2 \\ 13 & 5 & 6 \end{vmatrix} \\
& \begin{vmatrix} 0 & 0 & 0 & 1 \end{vmatrix}
\end{aligned}$$

$$= 5(-1)^{1+3} \begin{vmatrix} -8 & -1 \\ 13 & 5 \end{vmatrix}$$

$$= (5)(-27)$$

$$= -135$$

3.3 Properties of Determinants

- **Thm 3.5: (Determinant of a matrix product)**

$$\det (AB) = \det (A) \det (B)$$

- **Notes:**

(1) $\det (EA) = \det (E) \det (A)$

(2) $\det(A + B) \neq \det(A) + \det(B)$

(3)

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{22} + b_{22} & a_{22} + b_{22} & a_{23} + b_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ b_{21} & b_{22} & b_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

-
- Ex 1: (The determinant of a matrix product)

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & -2 \\ 3 & 1 & -2 \end{bmatrix}$$

Find $|A|$, $|B|$, and $|AB|$

Sol:

$$|A| = \begin{vmatrix} 1 & -2 & 2 \\ 0 & 3 & 2 \\ 1 & 0 & 1 \end{vmatrix} = -7 \quad |B| = \begin{vmatrix} 2 & 0 & 1 \\ 0 & -1 & -2 \\ 3 & 1 & -2 \end{vmatrix} = 11$$

$$AB = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & -2 \\ 3 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 8 & 4 & 1 \\ 6 & -1 & -10 \\ 5 & 1 & -1 \end{bmatrix}$$

$$\Rightarrow |AB| = \begin{vmatrix} 8 & 4 & 1 \\ 6 & -1 & -10 \\ 5 & 1 & -1 \end{vmatrix} = -77$$

■ Check:

$$|AB| = |A| |B|$$

- **Thm 3.6: (Determinant of a scalar multiple of a matrix)**

If A is an $n \times n$ matrix and c is a scalar, then

$$\det(cA) = c^n \det(A)$$

- **Ex 2:**

$$A = \begin{bmatrix} 10 & -20 & 40 \\ 30 & 0 & 50 \\ -20 & -30 & 10 \end{bmatrix}, \quad \begin{vmatrix} 1 & -2 & 4 \\ 3 & 0 & 5 \\ -2 & -3 & 1 \end{vmatrix} = 5$$

Find $|A|$.

Sol:

$$A = 10 \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & 5 \\ -2 & -3 & 1 \end{bmatrix} \Rightarrow |A| = 10^3 \begin{vmatrix} 1 & -2 & 4 \\ 3 & 0 & 5 \\ -2 & -3 & 1 \end{vmatrix} = (1000)(5) = 5000$$

- **Thm 3.7: (Determinant of an invertible matrix)**

A square matrix A is invertible (nonsingular) if and only if

$$\det(A) \neq 0$$

- **Ex 3: (Classifying square matrices as singular or nonsingular)**

$$A = \begin{bmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & 1 \end{bmatrix}$$

Sol:

$$|A| = 0 \quad \Rightarrow \quad A \text{ has no inverse (it is singular).}$$

$$|B| = -12 \neq 0 \quad \Rightarrow \quad B \text{ has an inverse (it is nonsingular).}$$

- **Thm 3.8: (Determinant of an inverse matrix)**

If A is invertible, then $\det(A^{-1}) = \frac{1}{\det(A)}$.

- **Thm 3.9: (Determinant of a transpose)**

If A is a square matrix, then $\det(A^T) = \det(A)$.

- **Ex 4:**

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 2 \\ 2 & 1 & 0 \end{bmatrix} \quad \text{(a) } |A^{-1}| = ? \quad \text{(b) } |A^T| = ?$$

Sol:

$$\begin{aligned} \because |A| &= \begin{vmatrix} 1 & 0 & 3 \\ 0 & -1 & 2 \\ 2 & 1 & 0 \end{vmatrix} = 4 & \quad \therefore |A^{-1}| = \frac{1}{|A|} = \frac{1}{4} \\ & & \quad |A^T| = |A| = 4 \end{aligned}$$

- Equivalent conditions for a nonsingular matrix:

If A is an $n \times n$ matrix, then the following statements are equivalent.

(1) A is invertible.

(2) $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $n \times 1$ matrix \mathbf{b} .

(3) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(4) A is row-equivalent to I_n

(5) A can be written as the product of elementary matrices.

(6) $\det(A) \neq 0$

-
- **Ex 5:** Which of the following system has a unique solution?

$$(a) \quad \quad \quad 2x_2 - x_3 = -1$$

$$3x_1 - 2x_2 + x_3 = 4$$

$$3x_1 + 2x_2 - x_3 = -4$$

$$(b) \quad \quad \quad 2x_2 - x_3 = -1$$

$$3x_1 - 2x_2 + x_3 = 4$$

$$3x_1 + 2x_2 + x_3 = -4$$

Sol:

(a) $A\mathbf{x} = \mathbf{b}$

$\because |A| = 0$

\therefore This system does not have a unique solution.

(b) $B\mathbf{x} = \mathbf{b}$

$\because |B| = -12 \neq 0$

\therefore This system has a unique solution.

3.4 Introduction to Eigenvalues

- **Eigenvalue problem:**

If A is an $n \times n$ matrix, do there exist $n \times 1$ nonzero matrices \mathbf{x} such that $A\mathbf{x}$ is a scalar multiple of \mathbf{x} ?

- **Eigenvalue and eigenvector:**

A : an $n \times n$ matrix

λ : a scalar

\mathbf{x} : a $n \times 1$ nonzero column matrix

$$\begin{array}{c} \text{Eigenvalue} \\ \downarrow \\ A\mathbf{x} = \lambda\mathbf{x} \\ \uparrow \quad \uparrow \\ \text{Eigenvector} \end{array}$$

(The fundamental equation for the eigenvalue problem)

- Ex 1: (Verifying eigenvalues and eigenvectors)

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \quad x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad x_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$Ax_1 = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 5x_1$$

Eigenvalue
↓
Eigenvalue
↑
Eigenvector

$$Ax_2 = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} = -1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} = (-1)x_2$$

Eigenvalue
↓
Eigenvalue
↑
Eigenvector

- **Question:**

Given an $n \times n$ matrix A , how can you find the eigenvalues and corresponding eigenvectors?

- **Note:**

$$Ax = \lambda x \implies (\lambda I - A)x = 0 \quad (\text{homogeneous system})$$

If $(\lambda I - A)x = 0$ has nonzero solutions iff $\det(\lambda I - A) = 0$

- **Characteristic equation of $A \in M_{n \times n}$:**

$$\det(\lambda I - A) = |(\lambda I - A)| = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0 = 0$$

-
- Ex 2: (Finding eigenvalues and eigenvectors)

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

Sol: Characteristic equation:

$$\begin{aligned} |\lambda I - A| &= \begin{vmatrix} \lambda - 1 & -4 \\ -2 & \lambda - 3 \end{vmatrix} \\ &= \lambda^2 - 4\lambda - 5 = (\lambda - 5)(\lambda + 1) = 0 \end{aligned}$$

$$\Rightarrow \lambda = 5, -1$$

Eigenvalues: $\lambda_1 = 5, \lambda_2 = -1$

$$(1) \lambda_1 = 5 \quad \Rightarrow (\lambda_1 \mathbf{I} - A)x = \begin{bmatrix} 4 & -4 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad t \neq 0$$

$$(2) \lambda_2 = -1 \quad \Rightarrow (\lambda_2 \mathbf{I} - A)x = \begin{bmatrix} -2 & -4 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2t \\ -t \end{bmatrix} = t \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad t \neq 0$$

-
- Ex 3: (Finding eigenvalues and eigenvectors)

$$A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

Sol: Characteristic equation:

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 & 2 \\ -1 & \lambda - 2 & -1 \\ 1 & 1 & \lambda \end{vmatrix} = (\lambda - 1)(\lambda + 1)(\lambda - 3) = 0$$

The eigenvalues : $\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = 3$

$$\lambda_1 = 1 \Rightarrow \lambda_1 \mathbf{I} - A = \begin{bmatrix} 0 & -2 & 2 \\ -1 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2t \\ t \\ t \end{bmatrix} \Rightarrow \text{eigenvectors: } t \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, t \neq 0$$

$$\lambda_2 = -1 \Rightarrow \lambda_2 \mathbf{I} - A = \begin{bmatrix} -2 & -2 & 2 \\ -1 & -3 & -1 \\ 1 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2t \\ -t \\ t \end{bmatrix} \Rightarrow \text{eigenvectors: } t \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, t \neq 0$$

$$\lambda_3 = 3 \Rightarrow \lambda_3 \mathbf{I} - \mathbf{A} = \begin{bmatrix} 2 & -2 & 2 \\ -1 & 1 & -1 \\ 1 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2t \\ -t \\ t \end{bmatrix} \Rightarrow \text{eigenvectors: } t \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}, t \neq 0$$

3.5 Applications of Determinants

- Matrix of cofactors of A :

$$[C_{ij}] = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix} \quad C_{ij} = (-1)^{i+j} M_{ij}$$

- Adjoint matrix of A :

$$\text{adj}(A) = [C_{ij}]^T = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

-
- **Thm 3.10: (The inverse of a matrix given by its adjoint)**

If A is an $n \times n$ invertible matrix, then

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$$

- **Ex:**

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\Rightarrow \det(A) = ad - bc$$

$$\text{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\begin{aligned} \Rightarrow A^{-1} &= \frac{1}{\det(A)} \text{adj}(A) \\ &= \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \end{aligned}$$

■ Ex 1 & Ex 2:

$$A = \begin{bmatrix} -1 & 3 & 2 \\ 0 & -2 & 1 \\ 1 & 0 & -2 \end{bmatrix} \quad (a) \text{ Find the adjoint of } A.$$

(b) Use the adjoint of A to find A^{-1}

Sol: $\because C_{ij} = (-1)^{i+j} M_{ij}$

$$\Rightarrow C_{11} = + \begin{vmatrix} -2 & 1 \\ 0 & -2 \end{vmatrix} = 4, \quad C_{12} = - \begin{vmatrix} 0 & 1 \\ 1 & -2 \end{vmatrix} = 1, \quad C_{13} = + \begin{vmatrix} 0 & -2 \\ 1 & 0 \end{vmatrix} = 2$$

$$C_{21} = - \begin{vmatrix} 3 & 2 \\ 0 & 2 \end{vmatrix} = 6, \quad C_{22} = + \begin{vmatrix} -1 & 2 \\ 1 & 2 \end{vmatrix} = 0, \quad C_{23} = - \begin{vmatrix} -1 & 3 \\ 1 & 0 \end{vmatrix} = 3$$

$$C_{31} = + \begin{vmatrix} 3 & 2 \\ -2 & 1 \end{vmatrix} = 7, \quad C_{32} = - \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = 1, \quad C_{33} = + \begin{vmatrix} -1 & 3 \\ 0 & -2 \end{vmatrix} = 2$$

\Rightarrow cofactor matrix of $A \Rightarrow$ adjoint matrix of A

$$[C_{ij}] = \begin{bmatrix} 4 & 1 & 2 \\ 6 & 0 & 3 \\ 7 & 1 & 2 \end{bmatrix} \quad \text{adj}(A) = [C_{ij}]^T = \begin{bmatrix} 4 & 6 & 7 \\ 1 & 0 & 1 \\ 2 & 3 & 2 \end{bmatrix}$$

\Rightarrow inverse matrix of A

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A) \quad \because \det(A) = 3$$

$$= \frac{1}{3} \begin{bmatrix} 4 & 6 & 7 \\ 1 & 0 & 1 \\ 2 & 3 & 2 \end{bmatrix} = \begin{bmatrix} \frac{4}{3} & 2 & \frac{7}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{2}{3} & 1 & \frac{2}{3} \end{bmatrix}$$

■ **Check:** $AA^{-1} = I$

- Thm 3.11: (Cramer's Rule)

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

⋮

$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

$$A\mathbf{x} = \mathbf{b} \quad A = [a_{ij}]_{n \times n} = [A^{(1)}, A^{(2)}, \dots, A^{(n)}] \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \neq 0$$

(this system has a unique solution)

$$A_j = \left[A^{(1)}, A^{(2)}, \dots, A^{(j-1)}, b, A^{(j+1)}, \dots, A^{(n)} \right]$$

$$= \begin{bmatrix} a_{11} & \cdots & a_{1(j-1)} & b_1 & a_{1(j+1)} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2(j-1)} & b_2 & a_{2(j+1)} & \cdots & a_{2n} \\ \vdots & & & \ddots & & & \vdots \\ a_{n1} & \cdots & a_{n(j-1)} & b_n & a_{n(j+1)} & \cdots & a_{nn} \end{bmatrix}$$

(i.e. $\det(A_j) = b_1 C_{1j} + b_2 C_{2j} + \cdots + b_n C_{nj}$)

$$\Rightarrow x_j = \frac{\det(A_j)}{\det(A)}, \quad j = 1, 2, \dots, n$$

■ Pf:

$$A \mathbf{x} = \mathbf{b}, \quad \det(A) \neq 0$$

$$\Rightarrow \mathbf{x} = A^{-1} \mathbf{b} = \frac{1}{\det(A)} \text{adj}(A) \mathbf{b}$$

$$= \frac{1}{\det(A)} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$= \frac{1}{\det(A)} \begin{bmatrix} b_1 C_{11} + b_2 C_{21} + \cdots + b_n C_{n1} \\ b_1 C_{12} + b_2 C_{22} + \cdots + b_n C_{n2} \\ \vdots \\ b_1 C_{1n} + b_2 C_{2n} + \cdots + b_n C_{nn} \end{bmatrix}$$

$$\begin{aligned}\Rightarrow x_j &= \frac{1}{\det(A)} (b_1 C_{1j} + b_2 C_{2j} + \cdots + b_n C_{nj}) \\ &= \frac{\det(A_j)}{\det(A)} \quad j = 1, 2, \dots, n\end{aligned}$$

- **Ex 4:** Use Cramer's rule to solve the system of linear equations.

$$\begin{aligned} -x + 2y - 3z &= 1 \\ 2x \quad \quad + z &= 0 \\ 3x - 4y + 4z &= 2 \end{aligned}$$

Sol:

$$\det(A) = \begin{vmatrix} -1 & 2 & -3 \\ 2 & 0 & 1 \\ 3 & -4 & 4 \end{vmatrix} = 10 \quad \det(A_1) = \begin{vmatrix} 1 & 2 & -3 \\ 0 & 0 & 1 \\ 2 & -4 & 4 \end{vmatrix} = 8$$

$$\det(A_2) = \begin{vmatrix} -1 & 1 & -3 \\ 2 & 0 & 1 \\ 3 & 2 & 4 \end{vmatrix} = -15, \quad \det(A_3) = \begin{vmatrix} -1 & 2 & 1 \\ 2 & 0 & 0 \\ 3 & -4 & 2 \end{vmatrix} = -16$$

$$x = \frac{\det(A_1)}{\det(A)} = \frac{4}{5} \quad y = \frac{\det(A_2)}{\det(A)} = \frac{-3}{2} \quad z = \frac{\det(A_3)}{\det(A)} = \frac{-8}{5}$$

Keywords in Section 3.5:

- matrix of cofactors :
- adjoint matrix :
- Cramer's rule :