Chapter 4
Vector Spaces

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4.1 Vectors in $\mathbb{R}^n$

- An ordered $n$-tuple:
  
  a sequence of $n$ real number $(x_1, x_2, \cdots, x_n)$

- $n$-space: $\mathbb{R}^n$
  
  the set of all ordered $n$-tuple
Ex:

\[ n = 1 \quad R^1 = 1\text{-space} \]
\[ \quad = \text{set of all real number} \]

\[ n = 2 \quad R^2 = 2\text{-space} \]
\[ \quad = \text{set of all ordered pair of real numbers} \ (x_1, x_2) \]

\[ n = 3 \quad R^3 = 3\text{-space} \]
\[ \quad = \text{set of all ordered triple of real numbers} \ (x_1, x_2, x_3) \]

\[ n = 4 \quad R^4 = 4\text{-space} \]
\[ \quad = \text{set of all ordered quadruple of real numbers} \ (x_1, x_2, x_3, x_4) \]
Notes:

(1) An $n$-tuple $(x_1, x_2, \ldots, x_n)$ can be viewed as a point in $\mathbb{R}^n$ with the $x_i$’s as its coordinates.

(2) An $n$-tuple $(x_1, x_2, \ldots, x_n)$ can be viewed as a vector $x = (x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$ with the $x_i$’s as its components.

Ex:
\[ u = (u_1, u_2, \cdots, u_n), \quad v = (v_1, v_2, \cdots, v_n) \] (two vectors in \( R^n \))

- **Equal:**
  \[ u = v \quad \text{if and only if} \quad u_1 = v_1, \ u_2 = v_2, \cdots, u_n = v_n \]

- **Vector addition (the sum of \( u \) and \( v \)):**
  \[ u + v = (u_1 + v_1, u_2 + v_2, \cdots, u_n + v_n) \]

- **Scalar multiplication (the scalar multiple of \( u \) by \( c \)):**
  \[ cu = (cu_1, cu_2, \cdots, cu_n) \]

- **Notes:**
  The sum of two vectors and the scalar multiple of a vector in \( R^n \) are called the standard operations in \( R^n \).
- **Negative:**
  \[-\mathbf{u} = (-u_1, -u_2, -u_3, \ldots, -u_n)\]

- **Difference:**
  \[\mathbf{u} - \mathbf{v} = (u_1 - v_1, u_2 - v_2, u_3 - v_3, \ldots, u_n - v_n)\]

- **Zero vector:**
  \[\mathbf{0} = (0, 0, \ldots, 0)\]

- **Notes:**
  (1) The zero vector \(\mathbf{0}\) in \(\mathbb{R}^n\) is called the **additive identity** in \(\mathbb{R}^n\).

  (2) The vector \(-\mathbf{v}\) is called the **additive inverse** of \(\mathbf{v}\).
Thm 4.2: (Properties of vector addition and scalar multiplication)

Let \( \mathbf{u}, \mathbf{v}, \) and \( \mathbf{w} \) be vectors in \( \mathbb{R}^n \), and let \( c \) and \( d \) be scalars.

1. \( \mathbf{u} + \mathbf{v} \) is a vector in \( \mathbb{R}^n \)
2. \( \mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u} \)
3. \( (\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w}) \)
4. \( \mathbf{u} + \mathbf{0} = \mathbf{u} \)
5. \( \mathbf{u} + (\mathbf{-u}) = \mathbf{0} \)
6. \( c \mathbf{u} \) is a vector in \( \mathbb{R}^n \)
7. \( c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v} \)
8. \( (c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u} \)
9. \( c(d\mathbf{u}) = (cd)\mathbf{u} \)
10. \( 1(\mathbf{u}) = \mathbf{u} \)
Ex 5: (Vector operations in $R^4$)

Let $u=(2, -1, 5, 0)$, $v=(4, 3, 1, -1)$, and $w=(-6, 2, 0, 3)$ be vectors in $R^4$. Solve $x$ for $x$ in each of the following.

(a) $x = 2u - (v + 3w)$

(b) $3(x+w) = 2u - v + x$

Sol: (a) $x = 2u - (v + 3w)$

$= 2u - v - 3w$

$= (4, -2, 10, 0) - (4, 3, 1, -1) - (-18, 6, 0, 9)$

$= (4 - 4 + 18, -2 - 3 - 6, 10 - 1 - 0, 0 + 1 - 9)$

$= (18, -11, 9, -8)$. 
(b) \[3(x + w) = 2u - v + x\]

\[3x + 3w = 2u - v + x\]

\[3x - x = 2u - v - 3w\]

\[2x = 2u - v - 3w\]

\[x = u - \frac{1}{2}v - \frac{3}{2}w\]

\[= (2,1,5,0) + (-2, \frac{-3}{2}, \frac{-1}{2}, \frac{1}{2}) + (9,-3,0, \frac{-9}{2})\]

\[= (9, \frac{-11}{2}, \frac{9}{2}, -4)\]
Thm 4.3: (Properties of additive identity and additive inverse)

Let $v$ be a vector in $\mathbb{R}^n$ and $c$ be a scalar. Then the following is true.

1. The additive identity is unique. That is, if $u+v=v$, then $u = 0$
2. The additive inverse of $v$ is unique. That is, if $v+u=0$, then $u = -v$
3. $0v=0$
4. $c0=0$
5. If $cv=0$, then $c=0$ or $v=0$
6. $-(-v) = v$
Linear combination:

The vector \( \mathbf{x} \) is called a **linear combination** of \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \), if it can be expressed in the form

\[
\mathbf{x} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \cdots + c_n \mathbf{v}_n \quad c_1, c_2, \ldots, c_n : \text{ scalar}
\]

**Ex 6:**

Given \( \mathbf{x} = (-1, -2, -2) \), \( \mathbf{u} = (0,1,4) \), \( \mathbf{v} = (-1,1,2) \), and \( \mathbf{w} = (3,1,2) \) in \( \mathbb{R}^3 \), find \( a, b, \) and \( c \) such that \( \mathbf{x} = a\mathbf{u} + b\mathbf{v} + c\mathbf{w} \).

**Sol:**

\[
\begin{align*}
-b + 3c &= -1 \\
a + b + c &= -2 \\
4a + 2b + 2c &= -2
\end{align*}
\]

\( \Rightarrow a = 1, \; b = -2, \; c = -1 \)

Thus \( \mathbf{x} = \mathbf{u} - 2\mathbf{v} - \mathbf{w} \)
Notes:

A vector \( \mathbf{u} = (u_1, u_2, \ldots, u_n) \) in \( \mathbb{R}^n \) can be viewed as:

- a \( 1 \times n \) row matrix (row vector): \( \mathbf{u} = [u_1, u_2, \ldots, u_n] \)

or

- a \( n \times 1 \) column matrix (column vector): \( \mathbf{u} = 
\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \)

(The matrix operations of addition and scalar multiplication give the same results as the corresponding vector operations)
Vector addition
\[ \mathbf{u} + \mathbf{v} = (u_1, u_2, \ldots, u_n) + (v_1, v_2, \ldots, v_n) \]
\[ = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n) \]
\[ \mathbf{u} + \mathbf{v} = [u_1, u_2, \ldots, u_n] + [v_1, v_2, \ldots, v_n] \]
\[ = [u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n] \]
\[ \mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix} \]

Scalar multiplication
\[ c\mathbf{u} = c(u_1, u_2, \ldots, u_n) \]
\[ = (cu_1, cu_2, \ldots, cu_n) \]
\[ c\mathbf{u} = c[u_1, u_2, \ldots, u_n] \]
\[ = [cu_1, cu_2, \ldots, cu_n] \]
\[ c\mathbf{u} = c \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} cu_1 \\ cu_2 \\ \vdots \\ cu_n \end{bmatrix} \]
Keywords in Section 4.1:

- ordered $n$-tuple: زوج نوني مرتب
- $n$-space: فضاء نوني
- equal: مساوي
- vector addition: جمع متجهي
- scalar multiplication: ضرب عددي
- negative: سالب
- difference: الفرق
- zero vector: متجه صفري
- additive identity: محايد جماعي
- additive inverse: معاكس جماعي
4.2 Vector Spaces

- Vector spaces:
  Let $V$ be a set on which two operations (vector addition and scalar multiplication) are defined. If the following axioms are satisfied for every $u$, $v$, and $w$ in $V$ and every scalar (real number) $c$ and $d$, then $V$ is called a vector space.

Addition:
(1) $u+v$ is in $V$
(2) $u+v=v+u$
(3) $u+(v+w)=(u+v)+w$
(4) $V$ has a zero vector $0$ such that for every $u$ in $V$, $u+0=u$
(5) For every $u$ in $V$, there is a vector in $V$ denoted by $-u$ such that $u+(-u)=0$
Scalar multiplication:

(6) \( cu \) is in \( V \).

(7) \( c(u + v) = cu + cv \)

(8) \( (c + d)u = cu + du \)

(9) \( c(du) = (cd)u \)

(10) \( 1(u) = u \)
Notes:

(1) A vector space consists of **four entities**:

   - A set of vectors, a set of scalars, and two operations

\[ V : \text{nonempty set} \]
\[ c : \text{scalar} \]
\[ + (u, v) = u + v : \text{vector addition} \]
\[ \cdot (c, u) = cu : \text{scalar multiplication} \]
\[ (V, +, \cdot) \text{ is called a vector space} \]

(2) \( V = \{0\} \): zero vector space
Examples of vector spaces:

1. **$n$-tuple space: $\mathbb{R}^n$**
   
   $$(u_1, u_2, \ldots, u_n) + (v_1, v_2, \ldots, v_n) = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n)$$
   
   $k(u_1, u_2, \ldots, u_n) = (ku_1, ku_2, \ldots, ku_n)$

2. **Matrix space: $V = M_{m \times n}$** (the set of all $m \times n$ matrices with real values)

   Ex: $(m = n = 2)$

   $$\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} u_{11} + v_{11} & u_{12} + v_{12} \\ u_{21} + v_{21} & u_{22} + v_{22} \end{bmatrix}$$

   $k\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix}$
(3) *n*-th degree polynomial space: $V = P_n(x)$

(the set of all real polynomials of degree $n$ or less)

$$p(x) + q(x) = (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n$$

$$kp(x) = ka_0 + ka_1x + \cdots + ka_nx^n$$

(4) Function space: $V = c(-\infty, \infty)$ (the set of all real-valued continuous functions defined on the entire real line.)

$$(f + g)(x) = f(x) + g(x)$$

$$(kf)(x) = kf(x)$$
Thm 4.4: (Properties of scalar multiplication)

Let \( v \) be any element of a vector space \( V \), and let \( c \) be any scalar. Then the following properties are true.

1. \( 0v = 0 \)
2. \( c0 = 0 \)
3. If \( cv = 0 \), then \( c = 0 \) or \( v = 0 \)
4. \( (-1)v = -v \)
Notes: To show that a set is not a vector space, you need only find one axiom that is not satisfied.

Ex 6: The set of all integer is not a vector space.

Pf: \[ 1 \in V, \frac{1}{2} \in R \]
\[ (\frac{1}{2})(1) = \frac{1}{2} \not\in V \quad \text{(it is not closed under scalar multiplication)} \]

Ex 7: The set of all second-degree polynomials is not a vector space.

Pf: Let \( p(x) = x^2 \) and \( q(x) = -x^2 + x + 1 \)
\[ \Rightarrow p(x) + q(x) = x + 1 \not\in V \]
(it is not closed under vector addition)
Ex 8:

$V = \mathbb{R}^2$ = the set of all ordered pairs of real numbers

vector addition: $(u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2)$

scalar multiplication: $c(u_1, u_2) = (cu_1, 0)$

Verify $V$ is not a vector space.

Sol:

$\therefore 1(1, 1) = (1, 0) \neq (1, 1)$

$\therefore$ the set (together with the two given operations) is not a vector space
Keywords in Section 4.2:

- vector space: فضاء متجهات
- $n$-space: فضاء نوني
- matrix space: فضاء مصفوفات
- polynomial space: فضاء متعددات الحدود
- function space: فضاء الدوال
4.3 Subspaces of Vector Spaces

- **Subspace:**
  \[
  (V, +, \cdot) \quad : \text{a vector space}
  \]
  \[
  W \neq \emptyset \quad \text{: a nonempty subset}
  \]
  \[
  W \subseteq V \quad \text{: a vector space (under the operations of addition and scalar multiplication defined in } V) \]

  \[
  \Rightarrow W \text{ is a subspace of } V
  \]

- **Trivial subspace:**

  Every vector space \( V \) has at least two subspaces.

  (1) Zero vector space \( \{0\} \) is a subspace of \( V \).

  (2) \( V \) is a subspace of \( V \).
Thm 4.5: (Test for a subspace)

If $W$ is a nonempty subset of a vector space $V$, then $W$ is a subspace of $V$ if and only if the following conditions hold.

1. If $u$ and $v$ are in $W$, then $u+v$ is in $W$.
2. If $u$ is in $W$ and $c$ is any scalar, then $cu$ is in $W$. 
Ex: Subspace of $R^2$

(1) $\{0\}$ $\quad 0 = (0, 0)$

(2) Lines through the origin

(3) $R^2$

Ex: Subspace of $R^3$

(1) $\{0\}$ $\quad 0 = (0, 0, 0)$

(2) Lines through the origin

(3) Planes through the origin

(4) $R^3$
Ex 2: (A subspace of $M_{2\times2}$)

Let $W$ be the set of all $2\times2$ symmetric matrices. Show that $W$ is a subspace of the vector space $M_{2\times2}$, with the standard operations of matrix addition and scalar multiplication.

Sol:

$\therefore W \subseteq M_{2\times2}$, $M_{2\times2}$ : vector spaces

Let $A_1, A_2 \in W$ \( (A_1^T = A_1, A_2^T = A_2) \)

\[ A_1, A_2 \in W \Rightarrow (A_1 + A_2)^T = A_1^T + A_2^T = A_1 + A_2 \quad (A_1 + A_2 \in W) \]

\[ k \in R, A \in W \Rightarrow (kA)^T = kA^T = kA \quad (kA \in W) \]

$\therefore W$ is a subspace of $M_{2\times2}$
Ex 3: (The set of singular matrices is not a subspace of $M_{2\times 2}$)

Let $W$ be the set of singular matrices of order 2. Show that $W$ is not a subspace of $M_{2\times 2}$ with the standard operations.

Sol:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in W, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in W$$

$$A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \notin W$$

$\therefore W_2$ is not a subspace of $M_{2\times 2}$
Ex 4: (The set of first-quadrant vectors is not a subspace of $R^2$)

Show that $W = \{(x_1, x_2) : x_1 \geq 0 \text{ and } x_2 \geq 0\}$, with the standard operations, is not a subspace of $R^2$.

Sol:

Let $u = (1, 1) \in W$

$\therefore (-1)u = (-1)(1, 1) = (-1, -1) \notin W$ \hspace{1cm} (not closed under scalar multiplication)

$\therefore W$ is not a subspace of $R^2$
Keywords in Section 4.3:

- subspace: فضاء جزئي
- trivial subspace: فضاء جزئي بسيط
Linear combination:

A vector \( \mathbf{v} \) in a vector space \( V \) is called a linear combination of the vectors \( \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k \) in \( V \) if \( \mathbf{v} \) can be written in the form

\[
\mathbf{v} = c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 + \ldots + c_k \mathbf{u}_k \quad \text{where} \quad c_1, c_2, \ldots, c_k : \text{ scalars}
\]
- **Ex 2-3: (Finding a linear combination)**

\[ \mathbf{v}_1 = (1,2,3) \quad \mathbf{v}_2 = (0,1,2) \quad \mathbf{v}_3 = (-1,0,1) \]

Prove (a) \( \mathbf{w} = (1,1,1) \) is a linear combination of \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \)

(b) \( \mathbf{w} = (1,-2,2) \) is not a linear combination of \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \)

**Sol:**

(a) \( \mathbf{w} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3 \)

\[
(1,1,1) = c_1 (1,2,3) + c_2 (0,1,2) + c_3 (-1,0,1) \\
= (c_1 - c_3, 2c_1 + c_2, 3c_1 + 2c_2 + c_3)
\]

\[
c_1 - c_3 = 1 \\
2c_1 + c_2 = 1 \\
3c_1 + 2c_2 + c_3 = 1
\]
\[
\begin{bmatrix}
1 & 0 & -1 & 1 \\
2 & 1 & 0 & 1 \\
3 & 2 & 1 & 1
\end{bmatrix}
\xrightarrow{\text{Guass–Jordan Elimination}}
\begin{bmatrix}
1 & 0 & -1 & 1 \\
0 & 1 & 2 & -1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\Rightarrow c_1 = 1 + t, \ c_2 = -1 - 2t, \ c_3 = t

(this system has infinitely many solutions)

\[ t=1 \]
\Rightarrow \mathbf{w} = 2\mathbf{v}_1 - 3\mathbf{v}_2 + \mathbf{v}_3 \]
(b)

\[ \mathbf{w} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3 \]

\[
\begin{bmatrix}
1 & 0 & -1 & 1 \\
2 & 1 & 0 & -2 \\
3 & 2 & 1 & 2
\end{bmatrix}
\begin{bmatrix}
1 \\
0 \\
1 \\
0
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & -1 & 1 \\
0 & 1 & 2 & -4 \\
0 & 0 & 0 & 7
\end{bmatrix}
\]

\[ \Rightarrow \text{this system has no solution (} \because 0 \neq 7) \]

\[ \Rightarrow \mathbf{w} \neq c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3 \]
- the span of a set: \( \text{span} (S) \)

If \( S=\{v_1, v_2, \ldots, v_k\} \) is a set of vectors in a vector space \( V \), then the span of \( S \) is the set of all linear combinations of the vectors in \( S \),

\[
\text{span}(S) = \{c_1 v_1 + c_2 v_2 + \cdots + c_k v_k \mid \forall c_i \in \mathbb{R}\}
\]

(the set of all linear combinations of vectors in \( S \))

- a spanning set of a vector space:

If every vector in a given vector space can be written as a linear combination of vectors in a given set \( S \), then \( S \) is called a spanning set of the vector space.
Notes:

\[
\text{span } (S) = V
\]

\[
\Rightarrow S \text{ spans (generates) } V
\]

\[
V \text{ is spanned (generated) by } S
\]

\[
S \text{ is a spanning set of } V
\]

Notes:

1. \( \text{span}(\emptyset) = \{0\} \)
2. \( S \subseteq \text{span}(S) \)
3. \( S_1, S_2 \subseteq V \)
   \[
   S_1 \subseteq S_2 \Rightarrow \text{span}(S_1) \subseteq \text{span}(S_2)
   \]
- Linear Independent (L.I.) and Linear Dependent (L.D.):

\[ S = \{ \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \} \] : a set of vectors in a vector space \( V \)

\[ c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \cdots + c_k \mathbf{v}_k = \mathbf{0} \]

(1) If the equation has only the trivial solution \((c_1 = c_2 = \cdots = c_k = 0)\), then \( S \) is called linearly independent.

(2) If the equation has a nontrivial solution (i.e., not all zeros), then \( S \) is called linearly dependent.
- **Notes:**

  1. $\emptyset$ is linearly independent
  2. $\mathbf{0} \in S \Rightarrow S$ is linearly dependent.
  3. $\mathbf{v} \neq \mathbf{0} \Rightarrow \{\mathbf{v}\}$ is linearly independent
  4. $S_1 \subseteq S_2$

     $S_1$ is linearly dependent $\Rightarrow S_2$ is linearly dependent

     $S_2$ is linearly independent $\Rightarrow S_1$ is linearly independent
Ex 8: (Testing for linearly independent)

Determine whether the following set of vectors in $\mathbb{R}^3$ is L.I. or L.D.

$S = \{(1, 2, 3), (0, 1, 2), (-2, 0, 1)\}$

Sol:

$c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3 = \mathbf{0} \quad \Rightarrow \quad 2c_1 + c_2 + c_3 = 0$

$\begin{bmatrix}
1 & 0 & -2 & 0 \\
2 & 1 & 0 & 0 \\
3 & 2 & 1 & 0
\end{bmatrix}
\xrightarrow{\text{Gauss-Jordan Elimination}}
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}$

$\Rightarrow c_1 = c_2 = c_3 = 0$ (only the trivial solution)

$\Rightarrow S$ is linearly independent
Ex 9: (Testing for linearly independent)

Determine whether the following set of vectors in $P_2$ is L.I. or L.D.

$$S = \{1+x - 2x^2, 2+5x - x^2, x+x^2\}$$

Sol:

$$c_1v_1 + c_2v_2 + c_3v_3 = 0$$

i.e. \(c_1(1+x - 2x^2) + c_2(2+5x - x^2) + c_3(x+x^2) = 0+0x+0x^2\)

\[
\begin{align*}
&\Rightarrow c_1 + 2c_2 = 0 \quad \Rightarrow \begin{bmatrix} 1 & 2 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 0 \end{bmatrix} \\
&\Rightarrow c_1 + 5c_2 + c_3 = 0 \quad \Rightarrow \begin{bmatrix} 1 & 5 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 5 & 1 \end{bmatrix} \\
&\Rightarrow 2c_1 - c_2 + c_3 = 0 \quad \Rightarrow \begin{bmatrix} -2 & -1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} -2 & -1 & 1 \end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
&\Rightarrow \text{This system has infinitely many solutions.} \\
&\quad \text{(i.e., This system has nontrivial solutions.)}
\end{align*}
\]

\[
\Rightarrow S \text{ is linearly dependent.} \quad \text{(Ex: } c_1=2, \ c_2=-1, \ c_3=3)\]
**Ex 10: (Testing for linearly independent)**

Determine whether the following set of vectors in 2×2 matrix space is L.I. or L.D.

\[
S = \left\{ \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \right\}
\]

**Sol:**

\[
c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3 = 0
\]

\[
c_1 \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\]
\[ \Rightarrow \quad 2c_1 + 3c_2 + c_3 = 0 \\
\quad c_1 \quad = 0 \\
\quad 2c_2 + 2c_3 = 0 \\
\quad c_1 + c_2 \quad = 0 \]

\[ \Rightarrow \quad \begin{bmatrix} 2 & 3 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \xrightarrow{\text{Gauss-Jordan Elimination}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

\[ \Rightarrow \quad c_1 = c_2 = c_3 = 0 \quad \text{(This system has only the trivial solution.)} \]

\[ \Rightarrow \quad S \text{ is linearly independent.} \]
Keywords in Section 4.4:

- linear combination: تركيب خطي
- spanning set: مجموعة المدى
- trivial solution: حل بسيط
- linear independent: الاستقلال الخطي
- linear dependent: الاعتماد الخطي