Chapter 4
Vector Spaces

4.1 Vectors in R^n

- An ordered *n*-tuple:

  a sequence of *n* real number $(x_1, x_2, \cdots, x_n)$

- *n*-space: R^n

  the set of all ordered n-tuple

- Ex:

$n = 1$ $\quad R^1$ = 1-space

= set of all real number

$n = 2$ $\quad R^2$ = 2-space

= set of all ordered pair of real numbers (x_1, x_2)

$n = 3$ $\quad R^3$ = 3-space

= set of all ordered triple of real numbers (x_1, x_2, x_3)

$n = 4$ $\quad R^4$ = 4-space

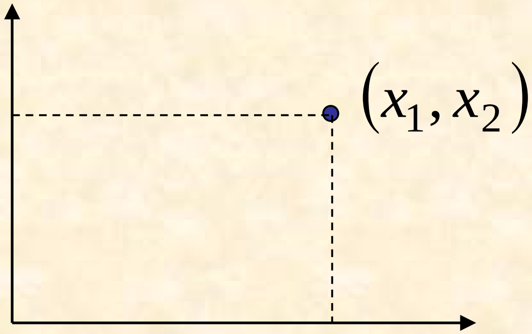
= set of all ordered quadruple of real numbers (x_1, x_2, x_3, x_4)

- Notes:

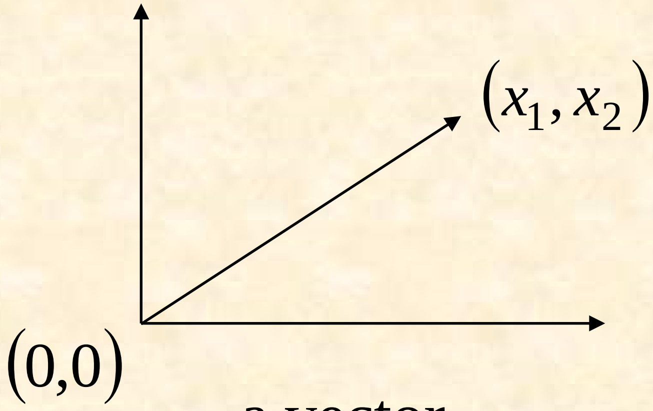
(1) An n-tuple $(x_1, x_2, \cdots, x_n)$ can be viewed as a point in R^n with the x_i's as its coordinates.

(2) An n-tuple $(x_1, x_2, \cdots, x_n)$ can be viewed as a vector $x = (x_1, x_2, \cdots, x_n)$ in R^n with the x_i's as its components.

- Ex:



a point



a vector

$\mathbf{u}=(u_1,u_2,\cdots,u_n),\ \mathbf{v}=(v_1,v_2,\cdots,v_n)$ (two vectors in R^n)

- Equal:

$\mathbf{u}=\mathbf{v}$ if and only if $u_1=v_1,\ u_2=v_2,\cdots,u_n=v_n$

- Vector addition (the sum of **u** and **v**):

$$\mathbf{u}+\mathbf{v}=(u_1+v_1,u_2+v_2,\cdots,u_n+v_n)$$

- Scalar multiplication (the scalar multiple of **u** by c):

$$c\mathbf{u}=(cu_1,cu_2,\cdots,cu_n)$$

- Notes:

The sum of two vectors and the scalar multiple of a vector in R^n are called the standard operations in R^n.

- Negative:

$$-\mathbf{u}=(-u_1,-u_2,-u_3,\ldots,-u_n)$$

- Difference:

$$\mathbf{u}-\mathbf{v}=(u_1-v_1,u_2-v_2,u_3-v_3,\ldots,u_n-v_n)$$

- Zero vector:

$$\mathbf{0}=(0,0,\ldots,0)$$

- Notes:

(1) The zero vector $\mathbf{0}$ in R^n is called the **additive identity** in R^n.

(2) The vector $-\mathbf{v}$ is called the **additive inverse** of $\mathbf{v}$.

- Thm 4.2: (Properties of vector addition and scalar multiplication)

Let **u**, **v**, and **w** be vectors in R^n , and let c and d be scalars.

(1) **u**+**v** is a vector in R^n

(2) **u**+**v** = **v**+**u**

(3) (**u**+**v**)+**w** = **u**+(**v**+**w**)

(4) **u**+**0** = **u**

(5) **u**+(–**u**) = **0**

(6) c**u** is a vector in R^n

(7) c(**u**+**v**) = c**u**+c**v**

(8) (c+d)**u** = c**u**+d**u**

(9) c(d**u**) = (cd)**u**

(10) 1(**u**) = **u**

- Ex 5: (Vector operations in R^4)

Let **u**=(2, – 1, 5, 0), **v**=(4, 3, 1, – 1), and **w**=(– 6, 2, 0, 3) be vectors in R^4. Solve **x** for x in each of the following.

(a) $\mathbf{x} = 2\mathbf{u} - (\mathbf{v} + 3\mathbf{w})$

(b) $3(\mathbf{x}+\mathbf{w}) = 2\mathbf{u} - \mathbf{v}+\mathbf{x}$

Sol: (a)

$$\begin{aligned}\mathbf{x} &= 2\mathbf{u} - (\mathbf{v} + 3\mathbf{w}) \\ &= 2\mathbf{u} - \mathbf{v} - 3\mathbf{w} \\ &= (4, -2, 10, 0) - (4, 3, 1, -1) - (-18, 6, 0, 9) \\ &= (4 - 4 + 18, -2 - 3 - 6, 10 - 1 - 0, 0 + 1 - 9) \\ &= (18, -11, 9, -8).\end{aligned}$$

$$
\begin{aligned}
\text{(b)}\quad 3(\mathbf{x}+\mathbf{w}) &= 2\mathbf{u}-\mathbf{v}+\mathbf{x} \\
3\mathbf{x}+3\mathbf{w} &= 2\mathbf{u}-\mathbf{v}+\mathbf{x} \\
3\mathbf{x}-\mathbf{x} &= 2\mathbf{u}-\mathbf{v}-3\mathbf{w} \\
2\mathbf{x} &= 2\mathbf{u}-\mathbf{v}-3\mathbf{w} \\
\mathbf{x} &= \mathbf{u}-\tfrac{1}{2}\mathbf{v}-\tfrac{3}{2}\mathbf{w} \\
&= (2,1,5,0)+\left(-2,\tfrac{-3}{2},\tfrac{-1}{2},\tfrac{1}{2}\right)+\left(9,-3,0,\tfrac{-9}{2}\right) \\
&= \left(9,\tfrac{-11}{2},\tfrac{9}{2},-4\right)
\end{aligned}
$$

- Thm 4.3: (Properties of additive identity and additive inverse)

Let **v** be a vector in R^n and c be a scalar. Then the following is true.

(1) The additive identity is unique. That is, if **u**+**v**=**v**, then **u** = **0**

(2) The additive inverse of **v** is unique. That is, if **v**+**u**=**0**, then **u** = –**v**

(3) 0**v**=**0**

(4) c**0**=**0**

(5) If c**v**=**0**, then c=0 or **v**=**0**

(6) –(– **v**) = **v**

- Linear combination:

The vector **x** is called a **linear combination** of $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n$,

if it can be expressed in the form

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n \qquad c_1, c_2, \cdots, c_n : \text{scalar}$$

- Ex 6:

Given **x** = (– 1, – 2, – 2), **u** = (0,1,4), **v** = (– 1,1,2), and **w** = (3,1,2) in R^3, find a, b, and c such that $\mathbf{x} = a\mathbf{u}+b\mathbf{v}+c\mathbf{w}$.

Sol:

$$\begin{aligned} -b + 3c &= -1 \\ a + b + c &= -2 \\ 4a + 2b + 2c &= -2 \end{aligned}$$

$$\Rightarrow a = 1,\ b = -2,\ c = -1$$

Thus $\mathbf{x} = \mathbf{u} - 2\mathbf{v} - \mathbf{w}$

- Notes:

A vector $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ in R^n can be viewed as:

a 1×n row matrix (row vector): $\mathbf{u} = [u_1,\ u_2, \cdots, u_n]$

or

a n×1 column matrix (column vector): $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$

（The matrix operations of addition and scalar multiplication give the same results as the corresponding vector operations)

Vector addition

$$\mathbf{u}+\mathbf{v}=(u_1,u_2,\cdots,u_n)+(v_1,v_2,\cdots,v_n)$$
$$=(u_1+v_1,u_2+v_2,\cdots,u_n+v_n)$$

$$\mathbf{u}+\mathbf{v}=[u_1,u_2,\cdots,u_n]+[v_1,v_2,\cdots,v_n]$$
$$=[u_1+v_1,u_2+v_2,\cdots,u_n+v_n]$$

$$\mathbf{u}+\mathbf{v}=\begin{bmatrix}u_1\\u_2\\\vdots\\u_n\end{bmatrix}+\begin{bmatrix}v_1\\v_2\\\vdots\\v_n\end{bmatrix}=\begin{bmatrix}u_1+v_1\\u_2+v_2\\\vdots\\u_n+v_n\end{bmatrix}$$

Scalar multiplication

$$c\mathbf{u}=c(u_1,\ u_2,\cdots,\ u_n)$$
$$=(cu_1,cu_2,\cdots,\ cu_n)$$

$$c\mathbf{u}=c[u_1,\ u_2,\cdots,\ u_n]$$
$$=[cu_1,cu_2,\cdots,\ cu_n]$$

$$c\mathbf{u}=c\begin{bmatrix}u_1\\u_2\\\vdots\\u_n\end{bmatrix}=\begin{bmatrix}cu_1\\cu_2\\\vdots\\cu_n\end{bmatrix}$$

Keywords in Section 4.1:

- ordered *n*-tuple：زوج نوني مرتب
- *n*-space：فضاء نوني
- equal：مساوي
- vector addition：جمع متجهي
- scalar multiplication：ضرب عددي
- negative：سالب
- difference：الفرق
- zero vector：متجه صفري
- additive identity：محايد جمعي
- additive inverse：معاكس جمعي

4.2 Vector Spaces

- Vector spaces:

  Let V be a set on which two operations (vector addition and scalar multiplication) are defined. If the following axioms are satisfied for every **u**, **v**, and **w** in V and every scalar (real number) c and d, then V is called a **vector space**.

  Addition:

  (1) $\mathbf{u}+\mathbf{v}$ is in V

  (2) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$

  (3) $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$

  (4) V has a zero vector **0** such that for every **u** in V, $\mathbf{u}+\mathbf{0}=\mathbf{u}$

  (5) For every **u** in V, there is a vector in V denoted by $-\mathbf{u}$ such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$

Scalar multiplication:

(6) $c\mathbf{u}$ is in V.

(7) $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$

(8) $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$

(9) $c(d\mathbf{u})=(cd)\mathbf{u}$

(10) $1(\mathbf{u})=\mathbf{u}$

- Notes:

(1) A vector space consists of four entities:

a set of vectors, a set of scalars, and two operations

V ： nonempty set

c ： scalar

$+(\mathbf{u},\mathbf{v}) = \mathbf{u}+\mathbf{v}:$ vector addition

$\bullet(c,\mathbf{u}) = c\mathbf{u}:$ scalar multiplication

$(V, \ +, \ \bullet)$ is called a vector space

(2) $V = \{\mathbf{0}\}$: zero vector space

- Examples of vector spaces:

(1) *n*-tuple space: R^n

$$(u_1,u_2,\cdots,u_n)+(v_1,v_2,\cdots,v_n)=(u_1+v_1,u_2+v_2,\cdots,u_n+v_n)$$ vector addition

$$k(u_1,u_2,\cdots,u_n)=(ku_1,ku_2,\cdots,ku_n)$$ scalar multiplication

(2) Matrix space: $V=M_{m\times n}$ (the set of all $m\times n$ matrices with real values)

Ex: ：$(m=n=2)$

$$\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}+\begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}=\begin{bmatrix} u_{11}+v_{11} & u_{12}+v_{12} \\ u_{21}+v_{21} & u_{22}+v_{22} \end{bmatrix}$$ vector addition

$$k\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}=\begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix}$$ scalar multiplication

(3) *n*-th degree polynomial space: $V = P_n(x)$

(the set of all real polynomials of degree *n* or less)

$$p(x)+q(x)=(a_0+b_0)+(a_1+b_1)x+\cdots+(a_n+b_n)x^n$$

$$kp(x)=ka_0+ka_1x+\cdots+ka_nx^n$$

(4) Function space: $V=c(-\infty,\infty)$ （the set of all real-valued continuous functions defined on the entire real line.)

$$(f+g)(x)=f(x)+g(x)$$

$$(kf)(x)=kf(x)$$

- Thm 4.4: (Properties of scalar multiplication)

Let $\mathbf{v}$ be any element of a vector space V, and let c be any scalar. Then the following properties are true.

(1) $0\mathbf{v} = \mathbf{0}$

(2) $c\mathbf{0} = \mathbf{0}$

(3) If $c\mathbf{v} = \mathbf{0}$, then $c = 0$ or $\mathbf{v} = \mathbf{0}$

(4) $(-1)\mathbf{v} = -\mathbf{v}$

- Notes: To show that a set is not a vector space, you need only find one axiom that is not satisfied.

- Ex 6: The set of all integer is not a vector space.

Pf:

$$1 \in V, \tfrac{1}{2} \in R$$

$$\underset{\text{scalar}}{\underset{\uparrow}{(\tfrac{1}{2})}}\underset{\text{integer}}{\underset{\uparrow}{(1)}} = \underset{\text{noninteger}}{\underset{\uparrow}{\tfrac{1}{2}}} \notin V$$

(it is not closed under scalar multiplication)

- Ex 7: The set of all second-degree polynomials is not a vector space.

Pf: Let $p(x) = x^2$ and $q(x) = -x^2 + x + 1$

$$\Rightarrow p(x) + q(x) = x + 1 \notin V$$

(it is not closed under vector addition)

- Ex 8:

  $V=R^2$=the set of all ordered pairs of real numbers

  vector addition: $(u_1, u_2)+(v_1, v_2)=(u_1+v_1, u_2+v_2)$

  scalar multiplication: $c(u_1, u_2)=(cu_1, 0)$

  Verify V is not a vector space.

Sol:

$\because 1(1,1)=(1,0)\neq(1,1)$

$\therefore$ the set (together with the two given operations) is not a vector space

Keywords in Section 4.2:

- vector space : فضاء متجهات
- *n*-space : فضاء نوني
- matrix space : فضاء مصفوفات
- polynomial space : فضاء متعددات الحدود
- function space : فضاء الدوال

4.3 Subspaces of Vector Spaces

- Subspace:

$(V,+,\bullet)$: a vector space

$\left.\begin{array}{l} W \neq \phi \\ W \subseteq V \end{array}\right\}$: a nonempty subset

$(W,+,\bullet)$: a vector space (under the operations of addition and scalar multiplication defined in V)

$\Rightarrow$ W is a subspace of V

- Trivial subspace:

Every vector space V has at least two subspaces.

(1) Zero vector space $\{\mathbf{0}\}$ is a subspace of V.

(2) V is a subspace of V.

- Thm 4.5: (Test for a subspace)

  If W is a nonempty subset of a vector space V, then W is a subspace of V if and only if the following conditions hold.

  (1) If $\mathbf{u}$ and $\mathbf{v}$ are in W, then $\mathbf{u}+\mathbf{v}$ is in W.

  (2) If $\mathbf{u}$ is in W and c is any scalar, then $c\mathbf{u}$ is in W.

- Ex: Subspace of R^2

  (1) $\{\mathbf{0}\}$ $\quad \mathbf{0}=(0,0)$

  (2) Lines through the origin

  (3) R^2

- Ex: Subspace of R^3

  (1) $\{\mathbf{0}\}$ $\quad \mathbf{0}=(0,0,0)$

  (2) Lines through the origin

  (3) Planes through the origin

  (4) R^3

- Ex 2: (A subspace of $M_{2\times 2}$)

Let W be the set of all 2×2 symmetric matrices. Show that W is a subspace of the vector space $M_{2\times 2}$, with the standard operations of matrix addition and scalar multiplication.

Sol:

$$\because W \subseteq M_{2\times 2} \qquad M_{2\times 2}: \text{vector sapces}$$

$$\text{Let } A_1, A_2 \in W \quad (A_1^T = A_1, A_2^T = A_2)$$

$$A_1 \in W, A_2 \in W \Rightarrow (A_1 + A_2)^T = A_1^T + A_2^T = A_1 + A_2 \qquad (A_1 + A_2 \in W)$$

$$k \in R, A \in W \Rightarrow (kA)^T = kA^T = kA \qquad (kA \in W)$$

$$\therefore W \text{ is a subspace of } M_{2\times 2}$$

- Ex 3: (The set of singular matrices is not a subspace of $M_{2\times2}$)

Let W be the set of singular matrices of order 2. Show that W is not a subspace of $M_{2\times2}$ with the standard operations.

Sol:

$$A=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\in W, B=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\in W$$

$$\therefore A+B=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\notin W$$

$\therefore W_2$ is not a subspace of $M_{2\times2}$

- Ex 4: (The set of first-quadrant vectors is not a subspace of R^2)

  Show that $W = \{(x_1, x_2) : x_1 \ge 0 \text{ and } x_2 \ge 0\}$, with the standard operations, is not a subspace of R^2.

Sol:

Let $\mathbf{u} = (1,1) \in W$

$$\because (-1)\mathbf{u} = (-1)(1,1) = (-1,-1) \notin W$$

(not closed under scalar multiplication)

$$\therefore W \text{ is not a subspace of } R^2$$

Keywords in Section 4.3:

- subspace：فضاء جزئي
- trivial subspace：فضاء جزئي بسيط

4.4 Spanning Sets and Linear Independence

- Linear combination:

A vector $\mathbf{v}$ in a vector space V is called a linear combination of the vectors $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$ in V if $\mathbf{v}$ can be written in the form

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots + c_k\mathbf{u}_k \qquad c_1, c_2, \cdots, c_k : \text{scalars}$$

- Ex 2-3: (Finding a linear combination)

$\mathbf{v}_1 = (1,2,3) \quad \mathbf{v}_2 = (0,1,2) \quad \mathbf{v}_3 = (-1,0,1)$

Prove (a) $\mathbf{w} = (1,1,1)$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$

(b) $\mathbf{w} = (1,-2,2)$ is not a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$

Sol:

(a) $\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$

$$(1,1,1) = c_1(1,2,3) + c_2(0,1,2) + c_3(-1,0,1)$$
$$= (c_1 - c_3, 2c_1 + c_2, 3c_1 + 2c_2 + c_3)$$

$$\Rightarrow \begin{aligned} c_1 \quad\quad - c_3 &= 1 \\ 2c_1 + c_2 \quad\quad &= 1 \\ 3c_1 + 2c_2 + c_3 &= 1 \end{aligned}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 1 \end{array}\right] \xrightarrow{\text{Guass}-\text{Jordan Elimination}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow c_1 = 1 + t\,,\ c_2 = -1 - 2t\,,\ c_3 = t$$

(this system has infinitely many solutions)

$$\overset{t=1}{\Rightarrow} \mathbf{w} = 2\mathbf{v}_1 - 3\mathbf{v}_2 + \mathbf{v}_3$$

(b)

$$\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & -2 \\ 3 & 2 & 1 & 2 \end{array}\right] \xrightarrow{\text{Guass–Jordan Elimination}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 0 & 7 \end{array}\right]$$

$$\Rightarrow \text{ this system has no solution } (\because 0 \neq 7)$$

$$\Rightarrow \mathbf{w} \neq c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

- the span of a set: span (*S*)

If $S=\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a set of vectors in a vector space V, then **the span of S** is the set of all linear combinations of the vectors in S,

$$span(S)=\{c_1\mathbf{v}_1+c_2\mathbf{v}_2+\cdots+c_k\mathbf{v}_k \mid \forall c_i \in R\}$$

(the set of all linear combinations of vectors in S)

- a spanning set of a vector space:

If every vector in a given vector space can be written as a linear combination of vectors in a given set S, then S is called **a spanning set** of the vector space.

- Notes:

$$\text{span}(S) = V$$

$\Rightarrow$ S spans (generates) V

V is spanned (generated) by S

S is a spanning set of V

- Notes:

(1) $span(\phi) = \{\mathbf{0}\}$

(2) $S \subseteq span(S)$

(3) $S_1, S_2 \subseteq V$

$$S_1 \subseteq S_2 \Rightarrow span(S_1) \subseteq span(S_2)$$

- Linear Independent (L.I.) and Linear Dependent (L.D.):

$S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\}$: a set of vectors in a vector space V

$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$

(1) If the equation has only the trivial solution ($c_1 = c_2 = \cdots = c_k = 0$)

then S is called linearly independent.

(2) If the equation has a nontrivial solution (i.e., not all zeros),

then S is called linearly dependent.

- Notes:

(1) ϕ is linearly independent

(2) $\mathbf{0} \in S \Rightarrow S$ is linearly dependent.

(3) $\mathbf{v} \neq \mathbf{0} \Rightarrow \{\mathbf{v}\}$ is linearly independent

(4) $S_1 \subseteq S_2$

S_1 is linearly dependent $\Rightarrow S_2$ is linearly dependent

S_2 is linearly independent $\Rightarrow S_1$ is linearly independent

- Ex 8: (Testing for linearly independent)

Determine whether the following set of vectors in R^3 is L.I. or L.D.

$$S = \{(1,2,3),(0,1,2),(-2,0,1)\}$$

$$\qquad \mathbf{v}_1 \qquad \mathbf{v}_2 \qquad \mathbf{v}_3$$

Sol:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0} \Rightarrow \begin{aligned} c_1 \qquad\quad - 2c_3 &= 0 \\ 2c_1 + c_2 + \qquad &= 0 \\ 3c_1 + 2c_2 + c_3 &= 0 \end{aligned}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \end{array}\right] \xrightarrow{\text{Gauss - Jordan Elimination}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\Rightarrow c_1 = c_2 = c_3 = 0 \quad (\text{only the trivial solution})$$

$$\Rightarrow S \text{ is linearly independent}$$

- Ex 9: (Testing for linearly independent)

Determine whether the following set of vectors in P_2 is L.I. or L.D.

$$S = \{\underset{\mathbf{v}_1}{1+x-2x^2},\ \underset{\mathbf{v}_2}{2+5x-x^2},\ \underset{\mathbf{v}_3}{x+x^2}\}$$

Sol:

$$c_1\mathbf{v}_1+c_2\mathbf{v}_2+c_3\mathbf{v}_3=\mathbf{0}$$

i.e. $c_1(1+x-2x^2)+c_2(2+5x-x^2)+c_3(x+x^2)=0+0x+0x^2$

$$\Rightarrow \begin{aligned} c_1+2c_2 \quad\quad &= 0 \\ c_1+5c_2+c_3 &= 0 \\ -2c_1-c_2+c_3 &= 0 \end{aligned} \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 1 & 5 & 1 & 0 \\ -2 & -1 & 1 & 0 \end{array}\right] \xrightarrow{\text{G.J.}} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 1 & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow$ This system has infinitely many solutions.
(i.e., This system has nontrivial solutions.)

$\Rightarrow$ S is linearly dependent. (Ex: $c_1=2$, $c_2=-1$, $c_3=3$)

- Ex 10: (Testing for linearly independent)

Determine whether the following set of vectors in 2×2 matrix space is L.I. or L.D.

$$S = \left\{ \underset{\mathbf{v}_1}{\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}}, \underset{\mathbf{v}_2}{\begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}}, \underset{\mathbf{v}_3}{\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}} \right\}$$

Sol:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$$

$$c_1\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} + c_2\begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix} + c_3\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{aligned} 2c_1 + 3c_2 + c_3 &= 0 \\ c_1 \qquad\qquad &= 0 \\ 2c_2 + 2c_3 &= 0 \\ c_1 + c_2 \qquad &= 0 \end{aligned}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 2 & 3 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right] \xrightarrow{\text{Gauss - Jordan Elimination}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow$ $c_1 = c_2 = c_3 = 0$ (This system has only the trivial solution.)

$\Rightarrow$ S is linearly independent.

Keywords in Section 4.4:

- linear combination : تركيب خطي
- spanning set : مجموعة المدى
- trivial solution : حل بسيط
- linear independent : الاستقلال الخطي
- linear dependent : الاعتماد الخطي