

**Question 1**

The velocity field within a laminar boundary layer is approximated by the following expression,

$$\vec{V} = \frac{AUy}{x^{1/2}} \hat{i} + \frac{AUy^2}{4x^{3/2}} \hat{j}$$

Where  $A = 141 \text{ m}^{-1/2}$  and  $U = 0.24 \text{ m/s}$ .

- Show that this velocity profile represents a possible incompressible flow.
- Calculate the acceleration of a fluid particle at  $(x, y) = (1, 1)$ .
- Determine the slope of the streamline through this point.

**Solution:**

- For incompressible flow we have,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial x} = -\frac{1}{2} \frac{AUy}{x^{3/2}}, \quad \frac{\partial v}{\partial y} = \frac{1}{4} \frac{AU(2y)}{x^{3/2}} = \frac{1}{2} \frac{AUy}{x^{3/2}}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{1}{2} \frac{AUy}{x^{3/2}} + \frac{1}{2} \frac{AUy}{x^{3/2}} = 0$$

Hence, this velocity field represents a possible incompressible flow.

- $\vec{a} = \frac{\partial \vec{V}}{\partial t} + \vec{V} \cdot \nabla \vec{V}$

$$a_x = u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y}$$

$$u \frac{\partial u}{\partial x} = -\left(\frac{AUy}{x^{1/2}}\right) \frac{1}{2} \frac{AUy}{x^{3/2}} = -\frac{1}{2} \left[\frac{AUy}{x}\right]^2$$

$$v \frac{\partial u}{\partial y} = \frac{AUy^2}{4x^{3/2}} \times \frac{AU}{x^{1/2}} = \frac{1}{4} \left[\frac{AUy}{x}\right]^2$$

$$a_x(1,1) = -\frac{1}{2} \left[\frac{AUy}{x}\right]^2 + \frac{1}{4} \left[\frac{AUy}{x}\right]^2 = -\frac{1}{4} \left[\frac{AUy}{x}\right]^2 = -286.3 \text{ m} \cdot \text{s}^{-2}$$

$$a_y = u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y}$$

$$u \frac{\partial v}{\partial x} = \frac{AUy}{x^{1/2}} \times \left(-\frac{3}{8} \frac{AUy^2}{x^{5/2}}\right) = -\frac{3}{8} (AU)^2 \left(\frac{y}{x}\right)^3$$

$$v \frac{\partial v}{\partial y} = \frac{1}{4} \frac{AUy^2}{x^{3/2}} \left(\frac{1}{4} \frac{AU(2y)}{x^{3/2}}\right) = \frac{1}{8} (AU)^2 \left(\frac{y}{x}\right)^3$$

$$a_y = -\frac{3}{8} (AU)^2 \left(\frac{y}{x}\right)^3 + \frac{1}{8} (AU)^2 \left(\frac{y}{x}\right)^3 = -\frac{1}{4} (AU)^2 \left(\frac{y}{x}\right)^3 = -\frac{1}{4} (141 \times 0.24) = -286.3 \text{ m} \cdot \text{s}^{-2}$$

$$\vec{a} = -286.3 \hat{i} - 286.3 \hat{j}$$

- The slope of the streamline passing through this point is,

$$\frac{dy}{dx} = \frac{v}{u} = \frac{\left[\frac{AUy^2}{4x^{3/2}}\right]}{\left[\frac{AUy}{x^{1/2}}\right]} = \frac{y}{4x}$$

$$\frac{dy}{dx}(1,1) = \frac{1}{4}$$

### Question 2

Assuming steady flow of water from the tank on the left to the tank on the right through the shown pipe in Figure 1, determine the flow rate in the connecting pipe. The dynamic viscosity of water is  $1 \times 10^{-3} \text{ kg/(m.s)}$  and the pipe is made of cast iron with  $e=0.26 \text{ mm}$ . Minor loss coefficients are shown on the figure as  $K_L$ .

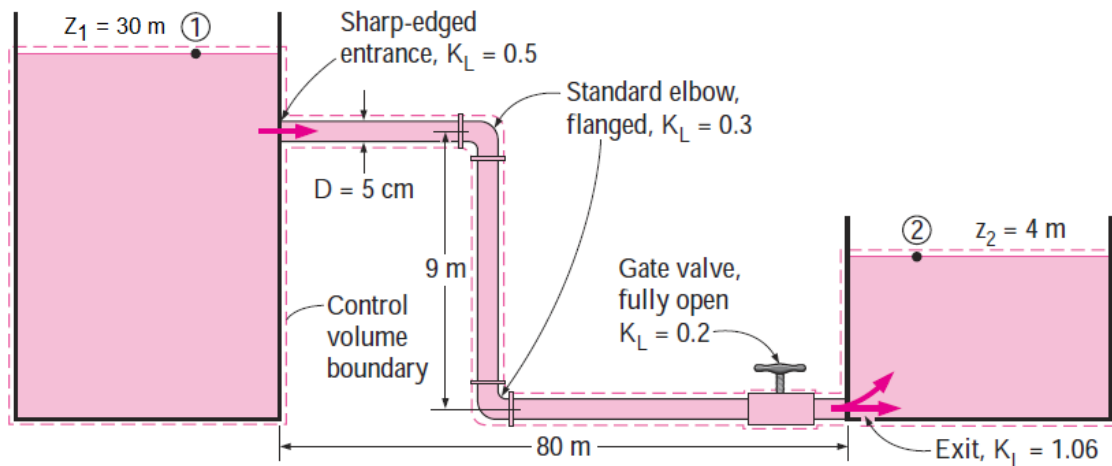


Figure 1, Question 2.

### Solution:

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2 + \frac{fL}{D} \frac{V^2}{2} + \sum_i K_i \frac{V^2}{2}$$

Since,  $p_1 = p_2 = p_{atm}$  and  $V_1, V_2 \approx 0$ , then

$$g(z_1 - z_2) = \frac{fL}{D} \frac{V^2}{2} + \sum_i K_i \frac{V^2}{2} = \left( \frac{fL}{D} + \sum_i K_i \right) \frac{V^2}{2}$$

$$L = 80 + 9 = 89 \text{ m}, D = 0.05 \text{ m}, \quad \sum_i K_i = 0.5 + 2 \times 0.3 + 0.2 + 1.06 = 2.36$$

$$V = \sqrt{\frac{2g(z_1 - z_2)}{\frac{fL}{D} + \sum_i K_i}} = \sqrt{\frac{2 \times 9.8 \times (30 - 4)}{1780f + 2.36}} = \sqrt{\frac{509.6}{1780f + 2.36}}$$

Since  $\frac{e}{D} = \frac{0.026}{5} = 0.0052$ , hence we can assume fully turbulent flow to obtain an estimate for  $f$  from Moody chart.

$$V = \sqrt{\frac{509.6}{1780f + 2.36}}$$

$f$	$V \text{ m/s}$	$Re$	$f_{new}$
0.03	3.023	151155	0.031
0.031	2.976	148798	0.0315
0.0315	2.953	147661	0.0315

$$V = 2.953 \text{ m/s}$$

$$Q = VA = \frac{V\pi D^2}{4} = 2.953 \times \pi \times \frac{0.05^2}{4} = 0.005798 \text{ m}^3/\text{s}$$

$$Q = 0.005798 \text{ m}^3/\text{s}$$

**Question 3**

- a. Write the Bernoulli's equation and state the conditions required for its validity.
- b. Consider the pressurized water tank shown in the figure below. The pressure above water is maintained at 300 kPa absolute while the atmospheric pressure is 100 kPa. Water is discharged through a 10 cm orifice at the bottom of the tank to the atmosphere. Assuming frictionless flow, calculate:
  - i. Exit flow velocity from the orifice
  - ii. Rate of water discharge from the tank in L/s.

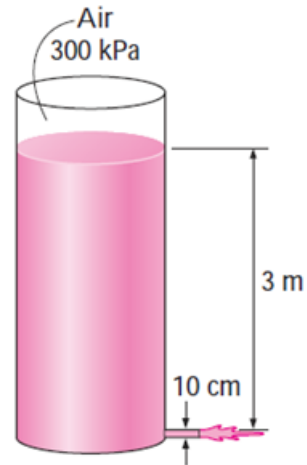


Figure 2, Question 3.

**Solution:**

a.  $\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{Constant}$

Steady Flow

Incompressible Fluid

Irrotational Flow

Along a streamline

b.  $\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2$

$p_1 = 300 \text{ kPa}, \quad p_2 = 100 \text{ kPa}, \quad z_1 = 3 \text{ m}, \quad z_2 = 0 \text{ m}, \quad V_1 \approx 0, \quad \rho = 10^3 \text{ kg/m}^3$

$$V_2 = \sqrt{\frac{2(p_1 - p_2)}{\rho} + 2gz_1} = \sqrt{\frac{2(300 - 100) \times 10^3}{10^3} + 2 \times 9.8 \times 3}$$

$$V_2 = 21.42 \text{ m/s}$$

Flow rate  $Q = V_2 A = \frac{\pi D^2}{4} V_2 = \pi \times \frac{0.1^2}{4} \times 21.42 = 0.168 \frac{\text{m}^3}{\text{s}} = 168 \frac{\text{L}}{\text{s}}$

#### Question 4

Consider the growth of the laminar boundary layer (LBL) along a flat plate. It is hypothesized that the boundary layer thickness at any location  $x$  along the plate depends on  $x$ , the fluid viscosity  $\mu$ , the fluid density  $\rho$  and upstream (freestream) velocity  $U$ , i.e.

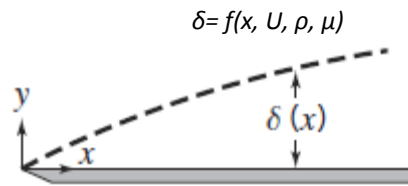


Figure 3, Question 4.

- a. Using dimensional analysis, show that:

$$\delta/x = \phi(Re_x)$$

where  $Re_x$  is the Reynolds number based on  $x$

- b. Using the boundary-layer momentum-integral equation with the following velocity profile

$$u/U = 3/2 \eta - 1/2 \eta^3$$

where  $\eta = y/\delta$ , show that:

- i. The above velocity profile satisfies 3 basic boundary conditions for the LBL.

ii.  $\delta/x = 4.64/\sqrt{Re_x}$

iii.  $C_f = 0.647/\sqrt{Re_x}$

where  $C_f = \tau_w / (1/2 \rho U^2)$

- c. For a flat plate with length of 1 m placed in 3 m/s parallel water stream ( $\mu=1 \times 10^{-3}$  kg/(m.s)), determine the boundary layer thickness and the shear stress at the end of the plate.

#### Solution:

- (a) Given  $\delta = f(x, U, \rho, \mu)$

Number of parameters  $m=5$

Dimensions of physical parameters

$\delta$	$x$	$U$	$\rho$	$\mu$
$L$	$L$	$L/T$	$M/L^3$	$M/L \cdot T$

Number of primary dimensions  $r=3$ ,

Then number of dimensionless groups is  $p=m-r=5-3=2$

Choose repeating parameters  $\rho, U$  and  $x$ . Then we have,

$$\Pi_1 = \phi(\Pi_2)$$

First group: choose  $\Pi_1 = \rho^a U^b x^c \delta$ , such that,

$$[\rho^a U^b x^c \delta] = M^0 L^0 T^0$$

$$[\rho^a][U^b][x^c][\delta] = M^0 L^0 T^0$$

$$\left(\frac{M^a}{L^{3a}}\right) \cdot \left(\frac{L^b}{T^b}\right) \cdot (L^c) \cdot (L) = M^0 L^0 T^0$$

Then we have,

$$a = 0, b = 0, c + 1 = 0 \rightarrow c = -1$$

Hence the first group is,

$$\Pi_1 = \frac{\delta}{x}$$

Second group: Choose  $\Pi_2 = \rho^d U^e x^f \mu$ , such that,

$$[\rho^d U^e x^f \mu] = M^0 L^0 T^0$$

$$\left(\frac{M^d}{L^{3d}}\right) \cdot \left(\frac{L^e}{T^e}\right) \cdot (L^f) \cdot \left(\frac{M}{LT}\right) = M^0 L^0 T^0$$

$$d + 1 = 0, \quad -3d + e + f - 1 = 0, \quad -e - 1 = 0$$

Hence, we have  $d = -1, e = -1$  and  $f = -1$ , hence the second group is equal to,

$$\Pi_2 = \frac{\mu}{\rho U x} = \frac{1}{Re_x}$$

Finally the required relations is,

$$\Pi_1 = g(\Pi_2)$$

$$\frac{\delta}{x} = \phi\left(\frac{\rho U d}{\mu}\right) = \phi(Re_x)$$

(b) The three basic boundary conditions for LBL are,

$$\begin{aligned} \text{a. At } \eta = 0, u = 0 & \quad u/U(0) = \frac{3}{2}(0) - \frac{1}{2}(0)^3 = 0 \\ \text{b. At } \eta = 1, u/U = 1 & \quad u/U(1) = \frac{3}{2}(1) - \frac{1}{2}(1)^3 = 1 \\ \text{c. At } \eta = 1, \frac{\partial(u/U)}{\partial\eta} = 0 & \quad \frac{\partial u/U}{\partial\eta}(1) = \frac{3}{2} - \frac{3}{2}(1)^2 = 0 \end{aligned}$$

(ii) Using the momentum-integral boundary layer equation

$$\frac{\tau_w}{\rho} = \frac{d}{dx}(U^2\theta) + \delta^* U \frac{dU}{dx}$$

Since for a flat plate  $\frac{dU}{dx} = 0$ ,  $\tau_w = \mu \frac{du}{dy}$  and

$$\begin{aligned} \theta &= \delta \int_0^1 \left(\frac{u}{U}\right) \left(1 - \frac{u}{U}\right) d\eta = \delta \int_0^1 \left(\frac{3}{2}\eta - \frac{1}{2}\eta^3\right) \left(1 - \frac{3}{2}\eta + \frac{1}{2}\eta^3\right) d\eta \\ \theta &= \delta \int_0^1 \left(\frac{3}{2}\eta - \frac{9}{4}\eta^2 + \frac{3}{4}\eta^4 - \frac{1}{2}\eta^3 + \frac{3}{4}\eta^4 - \frac{1}{4}\eta^6\right) d\eta \\ \theta &= \delta \int_0^1 \left(\frac{3}{2}\eta - \frac{9}{4}\eta^2 - \frac{1}{2}\eta^3 + \frac{3}{2}\eta^4 - \frac{1}{4}\eta^6\right) d\eta \\ \theta &= \delta \left[ \frac{3}{4}\eta^2 - \frac{3}{4}\eta^3 - \frac{1}{8}\eta^4 + \frac{3}{10}\eta^5 - \frac{1}{28}\eta^7 \right]_0^1 = \delta \left[ \frac{3}{4} - \frac{3}{4} - \frac{1}{8} + \frac{3}{10} - \frac{1}{28} \right] \\ \theta &= \frac{39}{280} \delta \end{aligned}$$

Then,

$$\begin{aligned} \frac{\tau_w}{\rho} = \frac{\mu}{\rho} \frac{du}{dy}_{y=0} &= \frac{d}{dx} \left( \frac{39}{280} U^2 \delta \right) = \frac{39 U^2}{280} \frac{d\delta}{dx} \\ \frac{du}{dy}_{y=0} &= \frac{3U}{2\delta} \end{aligned}$$

Hence,

$$\begin{aligned} \frac{3\mu U}{2\rho\delta} &= \frac{39U^2}{280} \frac{d\delta}{dx} \\ \delta \frac{d\delta}{dx} &= \frac{280}{26} \frac{\mu}{\rho U} \\ \delta d\delta &= \frac{280}{26} \frac{\mu}{\rho U} dx \\ \int_0^\delta \delta d\delta &= \int_0^x \frac{280}{26} \frac{\mu}{\rho U} d\tilde{x} \\ \frac{\delta^2}{2} &= \frac{280}{26} \frac{\mu}{\rho U} x \\ \frac{\delta}{x} &= \sqrt{\frac{280}{13} \frac{\mu}{\rho U x}} \approx \frac{4.64}{\sqrt{Re_x}} \end{aligned}$$

$$\begin{aligned} \text{(ii) } C_f &= \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{\mu \frac{3U}{2\delta}}{\frac{1}{2}\rho U^2} = \frac{3\mu}{\rho U \delta} = \frac{3\mu\sqrt{Re_x}}{4.64\rho U x} = \left(\frac{3}{4.64}\right) \frac{1}{\sqrt{Re_x}} \\ C_f &= \frac{0.647}{\sqrt{Re_x}} \end{aligned}$$

(c) For a 1m long flat plate  $U = 3 \frac{m}{s}$ ,  $\mu = 10^{-3} \frac{kg}{m \cdot s}$ ,  $\rho = \frac{10^3 kg}{m^3}$ ,  $Re_x = \frac{10^3 \times 3 \times 1}{10^{-3}} = 3 \times 10^6$

$$\begin{aligned} \delta &= \frac{4.64x}{\sqrt{Re_x}} = \frac{4.64 \times 1}{\sqrt{3 \times 10^6}} = 2.679 \times 10^{-3} m = 2.679 mm \\ \tau_w &= \mu \frac{3U}{2\delta} = 10^{-3} \times \frac{3}{2} \times \frac{3}{2.679 \times 10^{-3}} = 1.68 N/m^2 \end{aligned}$$

**Question 5:**

A gate is 1 m wide and 1.2 m tall and hinged at the bottom. On the right side, the gate holds back a 1-m deep body of water. On the left side a 5-cm diameter water jet hits the gate at a height of 1 m. Calculate,

- Calculate the force exerted on the gate due to the body of water on the right alone and its point of action
- Show that the force exerted by the jet on the gate is given by:  

$$F = \rho V^2 A_j$$
- Calculate the required jet speed to keep the gate vertical

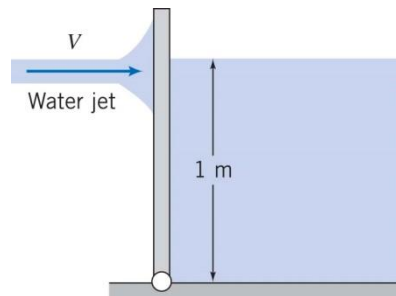


Figure 4, Question 5.

**Solution:**

- The force exerted by the body of water,  $F_R$ , is equal to,

$$F_R = p_c A = \rho \times g \times \frac{h}{2} \times h \times w$$

Where  $h = 1\text{ m}$ ,  $w = 1\text{ m}$ ,  $\rho = 10^3\text{ kg/m}^3$ ,  $g = 9.8\text{ m/s}^2$ .

$$F_R = 10^3 \times 9.8 \times \frac{1}{2} \times 1 \times 1 = 4900\text{ N}$$

The point of action of this force, measured from the water surface, is given by,

$$h' = y_c + \frac{\rho g I_{xx}}{p_c A} = \frac{h}{2} + \frac{\left(\frac{1}{12} w h^3\right)}{\frac{h}{2} \times h \times w} = \frac{h}{2} + \frac{h}{6} = \frac{2}{3} h = \frac{2}{3}\text{ m}$$

- The force exerted on the jet by the gate,  $F_j$ , is obtained by applying a control volume to the jet,

$$\frac{\partial}{\partial t} \iiint \rho V dV + \iint \rho V \vec{V} \cdot d\vec{A} = -\iint p d\vec{A} + F_j$$

Since the flow is steady, first term is zero, only one inlet and no exits in the x-direction, hence

$$-\rho V^2 A_j = F_j$$

Hence the force exerted on the jet by the gate is in the negative x-direction and the force exerted by the jet on the gate is equal to  $-F_j$ .

- For the gate to stay vertical the moment due to the force on the gate by the jet and the moment due to the force by the body of water on the right has to vanish at the hinge.

$$\rho V^2 A_j h = F_R (h - h')$$

$$V^2 = \frac{(F_R (h - h'))}{\rho A_j h} = \frac{\rho g \left(\frac{h}{2}\right) \left(\frac{h}{3}\right)}{\frac{\rho \pi D^2 h}{4}} = \frac{2g}{3\pi D^2 w}$$

$$V = \sqrt{\frac{2g}{3\pi D^2 w}} = \sqrt{\frac{2 \times 9.8}{3 \times \pi \times 0.05^2 \times 1}}$$

$$V = 28.84\text{ m/s}$$

**Useful Equations:**

$$\begin{aligned}\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) &= 0 \\ \frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla) \vec{V} &= -\frac{\nabla p}{\rho} + \nu \nabla^2 \vec{V} \\ \frac{D}{Dt}(\quad) &= \frac{\partial}{\partial t}(\quad) + \vec{V} \cdot \nabla(\quad) \\ \frac{p_1}{\rho} + \frac{\bar{V}_1^2}{2} + gz_1 &= \frac{p_2}{\rho} + \frac{\bar{V}_2^2}{2} + gz_2 + h_{l_T} \\ h_{l_T} &= f \frac{L}{D} \frac{\bar{V}^2}{2} + \sum K_L \frac{V^2}{2} \\ \delta^* &= \int_0^\infty \left(1 - \frac{u}{U}\right) dy \\ \theta &= \int_0^\infty \frac{u}{U} \left(1 - \frac{u}{U}\right) dy \\ \frac{\tau_w}{\rho} &= \frac{d}{dx} (U^2 \theta) + \delta^* U \frac{dU}{dx}\end{aligned}$$

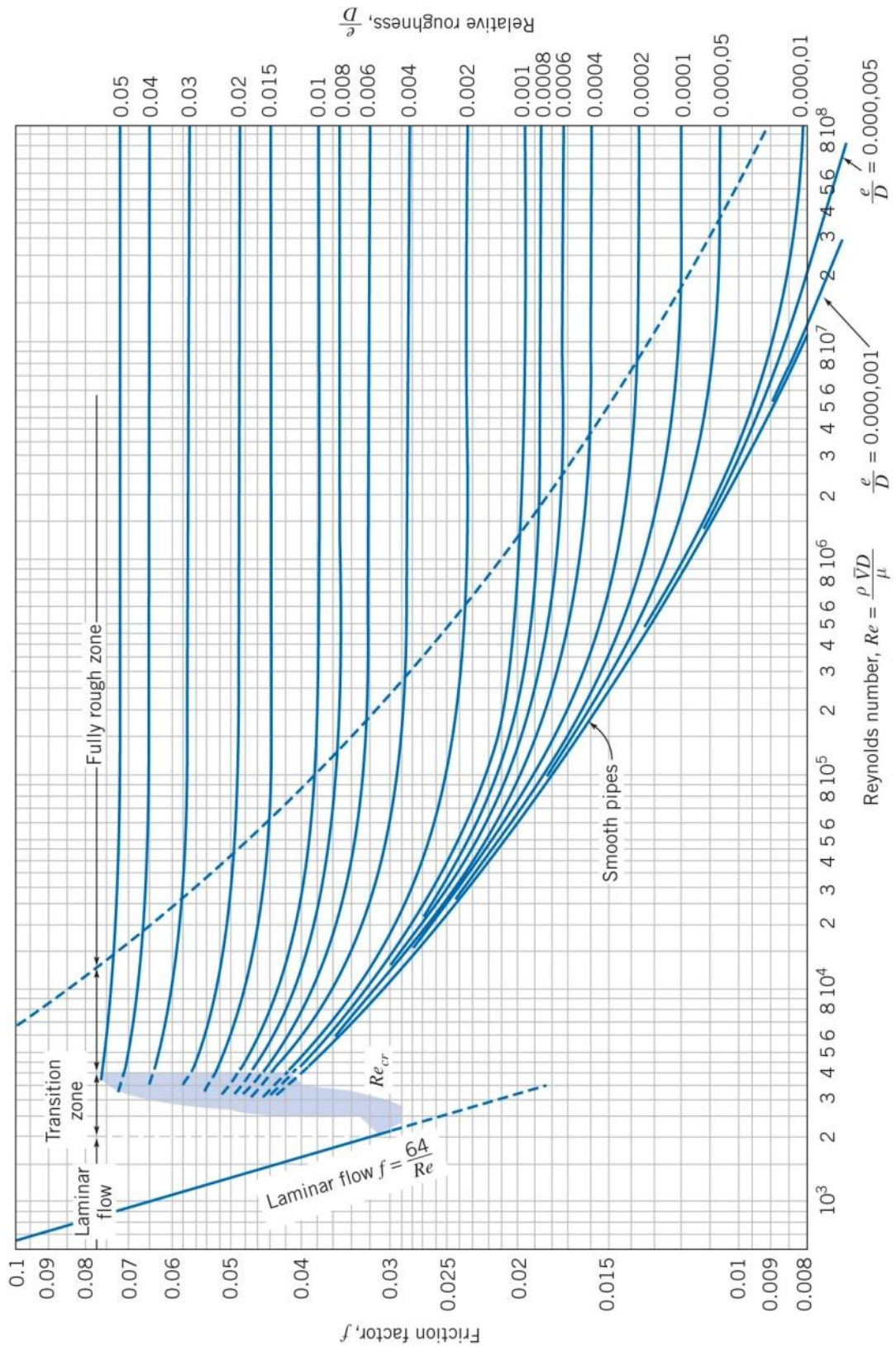


Figure 5, Moody Chart, Fox and McDonald, Eighth Edition, Wiley