Merton Problem for a Discrete Market in an Infinite Horizon and with Frictions

S. Ounaies[a], J-M. Bonnisseau[b], S. Chebbi[c,*] and H.M. Soner[d]

[a] Paris School of Economics, University of Paris 1, Panthéon Sorbonne, France & Department of Mathematics, College of Science, University El-Manar, Tunis, Tunisia. E-mail: ou-cynda@hotmail.com

[b] Paris School of Economics, University of Paris 1, Panthéon Sorbonne, CNRS, CES. M.S.E. 106 Boulevard de l'Hôpital, 75647 Paris cedex 13, France. E-mail: jmbonnisseau@univ-paris1.fr

[c] King Saud University, College of Science, Department of Mathematics, College of Science, King Saud University, Box 2455, Riyadh 11451, Saudi Arabia. E-mail: schebbi@ksu.edu.sa

[d] Department of Mathematics, Swiss Federal Institute of Technology (ETH) Zurich and Swiss Finance Institute. E-mail: hmsoner@ethz.ch.

ABSTRACT.
We investigate the problem of optimal investment and consumption of Merton in the case of discrete markets in an infinite horizon. We suppose that there is frictions in the markets due to loss in trading. These frictions are modeled through nonlinear penalty functions and the classical transaction cost and liquidity models are included in this formulation. In this context, the solvency region is defined taking into account this penalty function and every investigator have to maximize his utility, that is derived from consumption, in this region. We give the dynamic programming of the model and we prove the existence and uniqueness of the value function.

Key Words and phrases: Merton problem, discrete market, infinite horizon, market frictions, after liquidation value, dynamic programming, value function.

---

*Corresponding author: King Saud University, College of Science, Department of Mathematics, P.O. Box 2455, Riyadh 11451, Saudi Arabia. Tel.: +96614676308; Fax: +96614676512, e-mail: schebbi@ksu.edu.sa.

Research supported by the NPSTI strategic technologies program in the Kingdom of Saudi Arabia, project number 12-MAT2703-02.

1 Introduction

In a very known paper appeared in 1971, Merton developed and modeled the problem of optimal investment and consumption in continuous time. Since it appear, this problem has been extensively investigated in the literature and extended in many directions, we refeer to the book of Karatzas and Shreve [9] for some extensions in this way. Recently, Chebbi and Soner in [1] consider the model of Merton when there is frictions in the market due to loss in trading. This paper is a study in this direction and the markets considered are discrete in infinite horizon.

In the literature, we can find several types of market friction. The first one that receive the most attention is the proportional transaction costs, first introduced and studied in the context of Merton problem by Magill and Constantinides [11] and later by Constantinides [2]. Recently, another concept of friction has been introduced by Cetin, Jarrow and Protter [3] for an illiquid market. Our concept of friction in this paper will be formulated through a convex penalty function g in a discrete market considered in an infinite horizon. This formulation will included both the function of proportional costs considered in [11] and the one considered for an illiquid market with no bid and ask spread [3]. The discrete time formulation of Merton problem was firstly developed by Jouini and Kallal [8] and in our context, the advantage of this type of formulation is that we can give a uniform approach that cove both the two principal type of frictions, i.e. proportional costs and illiquid marckets, while in continuous time one have to distinguish the case when g is differentiable at the origin or not.

In section 2, we extend the model of Merton with friction studied in [1] to the case of an infinite horizon. Using the penalty function, we give the dynamics of the cash and stock position.

In section 3, we study the optimal investment and the consumption problem of Merton. This problem is formulated as an optimization problem in which every investor has to maximize his expected utility function under a constraint condition defined by a solvency region. The utility function is derived from consumptions and the solvency region is defined through a natural condition concerning the non negativeness of what we call the after liquidation value, when an investor is forced to liquidate all stock positions. Then, we prove the dynamic programming of the model and by using a fixed point approach, we deduce the existence and uniqueness of the value function.

2 The Model

We consider a discrete market model in an infinite horizon. We suppose that the market is with a money market account and N risky assets and we assume that the money market account pays a return of fraction $r > 0$ of the invested amount. The risky assets, called the stocks, provide a random return of $R = (R_k)_{k\geq 1}$ with values in $[-1,\infty)^N$. The returns are supposed to be identically and independently distributed over time. We let μ be the common probability measure of $R_k's$, which is supposed to be finite on $\mathbb{R}^N$. We consider the probability space $(\Omega, \mathscr{F}, \mathbb{P})$ where $\Omega = (\mathbb{R}^N)^\infty$ denotes the space of events $(\omega_k)_{k\geq 1}$ such that for all $k \in \mathbb{N}^*$, $\omega_k \in \mathbb{R}^N$. For $k \in \mathbb{N}^*$, we define the canonical mapping process $B_k(\omega) = \omega_k$, $k \geq 1$, $\omega \in \Omega$. We denote by $\mathscr{F}_k = \sigma(B_s; s \in \{1,2,...,k\})$ the σ-field generated by the canonical map, which represents the information that the investor has at any time k. We set $\mathscr{F}_\infty = \sigma(\bigcup_{k\in\mathbb{N}} \mathscr{F}_k)$, with $\mathscr{F}_0 = \{\emptyset, \Omega\}$ is the trivial σ-algebra. Let $\mathbb{P}$ the product probability measure given by

$$\mathbb{P}(\{\omega \in \Omega, \omega_k \in A_k, k \geq 1\}) = \prod_{k\geq 1} \mu(A_k).$$

Now, we let the return vector at time k given by $R_k(\omega) = B_k(\omega) = \omega_k$, $k \in \mathbb{N}^*$. Then $R_k's$ are $\mathscr{F}_k$-measurable, hence $R = (R_k)_{k\geq 1}$ is an $(\mathbb{R})^N$-valued, $\mathbb{F}$-adapted process. The connection between the stock process $S = (S_k)_{k\geq 1}$, where S_k^i is the ith stock at time k, and the return process R is simply given by

$$S_k^i = S_0^i \prod_{j\geq 1} [1 + R_j^i] \Longleftrightarrow R_k^i = \frac{S_k^i - S_{k-1}^i}{S_{k-1}^i}, \quad i = 1, \cdots, N.$$

where S_0^i is the initial stock value. Since $R_j^i \geq -1$, S is an $(\mathbb{R}^+)^N$-valued $\mathscr{F}$-adapted process.

The portfolio position of the investor is an $\mathscr{F}$-adapted, $\mathbb{R} \times (\mathbb{R}^+)^N$-valued process (x, y) and it has the following interpretation,

$x = (x_k)_{k\geq 1}$ = process of money invested in the money market account at any time k.

$y = (y_k^i)_{k\geq 1}$ = process of money invested in the i-th stock at any time k prior to the portfolio adjustment.

For $k \geq 1$, let $z = (z_k)_{k\geq 1}$ be the process of the number of shares of stock held by the investor at time k prior to the portfolio adjustment. Hence, z_k is $\mathscr{F}_{k-1}$-measurable and z in an $\mathscr{F}$-predictable process with values in $\mathbb{R}^N$. Moreover,

$$y_k^i = z_k^i S_k^i, \quad i = 1, \cdots, N; \quad k \in \mathbb{N}^*.$$

In our model, we assume that the market is with friction since trading results in a loss is a certain small percentage of the traded dollar amount:

$$\alpha_k^i := S_k^i \Delta_k z^i = S_k^i \, (z_{k+1}^i - z_k^i), \quad i = 1, \ldots, N, \;\; k \geq 1. \tag{2.1}$$

We thus suppose that there is a penalty function $g : \mathbb{R}^N \to [0, \infty)$, in the market which is assumed to be convex with $g(0) = 0$ and $g \geq 0$.

In this context, the dynamics for the cash position will be the following:

$$x_{k+1} = (x_k - \langle \alpha_k, 1 \rangle - g(\alpha_k) - c_k)\,(1 + r), \qquad k \geq 1, \tag{2.2}$$

where the non-negative, $\mathscr{F}$-adapted process c is the *consumption* of the investor, $\langle \cdot, \cdot \rangle$ denotes the usually inner product in $\mathbb{R}^N$.

Specific examples of a loss function in the literature are

$$g(\alpha) = \sum_{i=1}^{N} \lambda_i |\alpha^i|, \quad \text{or} \quad g(\alpha) = \sum_{i=1}^{N} \lambda_i (\alpha^i)^2,$$

where λ^i's are given non-negative (small) constants. The first of the above example corresponds to the classical example of the proportional costs [5, 6, 8, 11, 13]. The second, however, is a model of illiquidity [3, 4, 7]. origin. The main difference between the two examples is the differentiability at the origin. Indeed, a non-differentiability of g at the origin corresponds to a proportional transaction costs, or equivalently the existence of a bid-ask spread in the market.

The dynamics of the y process is the classical one defined for $k \geq 1$ by:

$$
\begin{aligned}
y_{k+1}^i &= y_k^i + \left[z_{k+1}^i S_{k+1}^i - z_k^i S_k^i\right] \\
&= y_k^i + S_k^i \left[z_{k+1}^i - z_k^i\right] + z_{k+1}^i \left[S_{k+1}^i - S_k^i\right] \\
&= y_k^i + \alpha_k^i + S_k^i z_{k+1}^i \left(\frac{S_{k+1}^i - S_k^i}{S_k^i} \right) \\
&= y_k^i + \alpha_k^i + \left[S_k^i (z_{k+1}^i - z_k^i) + z_k^i S_k^i\right] \; R_{k+1}^i \\
&= y_k^i + \alpha_k^i + \left(\alpha_k^i + y_k^i\right) \; R_{k+1}^i \\
&= \left(y_k^i + \alpha_k^i\right) \; (1 + R_{k+1}^i). \qquad (2.3)
\end{aligned}
$$

Notice that the dynamics of the state variables (x,y) in (2.2)-(2.3) are given only through the process α and not z. Hence, in whatever follows, we use the $\mathscr{F}$-adapted process α instead of z.

We also note that the mark-to-market value

$$
\omega_k := x_k + \langle y_k, 1 \rangle = x_k + \sum_{i=1}^{N} y_k^i
$$

satisfies the equation

$$
\begin{aligned}
\omega_{k+1} &= \omega_k + r x_k + [\alpha_k + y_k] \cdot R_k - \alpha_k \cdot \vec{1} r - c_k (1+r) - g(\alpha_k)(1+r) \\
&= \omega_k \left[1 + r + \pi_k \cdot (R_{k+1} - r)\right] - c_k (1+r) - g(\alpha_k)(1+r),
\end{aligned}
$$

where $\pi_k^i := [\alpha_k^i + y_k^i]/w_k$ is the fraction of the mark-to-market value invested in the stock after the portfolio adjustment. Indeed, this is the classical wealth equation when there is no friction, i.e, when $g \equiv 0$.

3 Solvency Region

It is well known that the optimal investment and consumptions type problem of Merton require a lower bound on the wealth like variables, see [9]. Otherwise, one may easily obtain non intuitive trivial results as consumption with no bound would be admissible. In this context, an appropriate notion is to require the mark-to-market value of the portfolio to be non-negative. In our model of markets with frictions, an admissibility type condition can be defined by taking into account the penalty function.

For a portfolio position $(x,y) \in \mathbb{R} \times (\mathbb{R}^+)^N$, we define the *cash value* of the portfolio at any time k by:

$$
L(x_k, y_k) = x_k + y_k - g(\alpha_k) \text{ and } \mathbb{L} = \{(x,y) \in \mathbb{R} \times \mathbb{R}^N : L(x,y) > 0\}, \qquad (3.1)
$$

and the *after-liquidation value* is defined simply as the cash value of the position after the investor is forced to liquidate (i.e., sell or close) all stock positions. Due to the loss function postulated in (2.2) this value differs from the mark-to-market value defined in the previous subsection. Indeed, using the idea behind (2.2), with $z_k = y_k / S_k$, $z_{k+1} = 0$, we obtain $\alpha_k = -y_k$ and the solvency condition is simply as follows:

Definition 1 *A* control process $\nu := (c,\alpha)$ *consists of a non-negative, $\mathscr{F}$-adapted consumption process c and an $\mathbb{R}^N$-valued, $\mathscr{F}$-adapted portfolio adjustment process α. We say that a control process $\nu = (c,\alpha)$ is* admissible *with initial position $(x,y) \in \overline{\mathbb{L}}$, if the solution $(x_k, y_k)_{k\geq 1}$ corresponding to* (2.2)-(2.3) *with initial data $x_0 = x$, $y_0 = y$ and controls (c,α) satisfies*

$$L(x_k,y_k) = x_k + \langle y_k, 1\rangle - g(-y_k) \geq 0, \quad \Longleftrightarrow \quad (x_k,y_k) \in \overline{\mathbb{L}}, \quad \forall k \geq 1,$$

$\mathbb{P}$-almost surely. We denote by $\mathbb{A}(x,y)$ the set of all admissible controls.

□

In the general context, we simply define

$$\mathbb{U}(x,y) := \left\{(c,\alpha) \in \mathbb{R}^+ \times \mathbb{R}^N \ : \ L(x_k,y_k) \geq 0, \ \mathbb{P}-a.s.\right\}. \tag{3.2}$$

We may rewrite the admissibility criterion using the sets $\mathbb{U}(x,y)$ as well. For future reference, we record this simple connection,

$$(c,\alpha) \in \mathbb{A}(x,y) \quad \Longleftrightarrow \quad (c_k,\alpha_k) \in \mathbb{U}(x_k,y_k), \quad \forall\ k \geq 1, \tag{3.3}$$

where (x_k,y_k) is the solution of (2.2)-(2.3).

Lemma 1 *For any $(x,y) \in \overline{\mathbb{L}}$, the admissible class of controls $\mathbb{A}(x,y)$) (and also $\mathbb{U}(x,y)$) is nonempty and convex.*

Proof.
To prove that $\mathbb{A}(x,y) \neq \emptyset$, take as a control process: $c \equiv 0$, $\alpha_0 = -y$ and $\alpha_k = 0$ for all $k \geq 1$. Then, the solution of (2.2)-(2.3) at time $k \geq 1$ is given by $y_k = 0$ and

$$x_k = (x + \langle y, 1\rangle - g(-y))(1+r)^k.$$

Then,

$$L(x_k,y_k) = x_k = (x + \langle y, 1\rangle - g(-y))(1+r)^k \geq 0,$$

since $(x,y) \in \bar{\mathbb{L}}$ is equivalent to $x + \langle y, 1\rangle - g(-y)) \geq 0$. So $\mathbb{U}(x,y)$ (resp. $\mathbb{A}(x,y)$) is nonempty.

Now we want to show that $\mathbb{A}(x,y)$ is convex. Take $(c^i,\alpha^i) \in \mathbb{A}(x^i,y^i)$, for $i = 1,2$, i.e. $(c^i_k,\alpha^i_k) \in \mathbb{U}(x^i_k,y^i_k)$ for $i = 1,2$ and $k \geq 1$. For $\lambda \in [0,1]$, we note by $\bar{c}_k = \lambda c^1_k + (1-\lambda)c^2_k$ and similarly $\bar{\alpha}_k$, $\bar{x}_k$, $\bar{y}_k$. We have:

$$\begin{aligned} L(\bar{x}_k,\bar{y}_k) &= \bar{x}_k + \bar{y}_k - g(\bar{\alpha}_k) \\ &= \lambda(x^1_k + y^1_k) + (1-\lambda)(x^2_k + y^2_k) - g(\bar{\alpha}_k) \\ &\geq \lambda g(\alpha^1_k) + (1-\lambda)g(\alpha^2_k) - g(\bar{\alpha}_k) \\ &\geq 0 \end{aligned}$$

since g is convex and $(x^i_k,y^i_k) \in \bar{\mathbb{L}}$ for $i = 1,2$ and $k \geq 1$.

□

Now for $\delta > 0$ and $I \subset \{1,\ldots,N\}$ define the set

$$\Omega^{\delta,I} := \{R^i_1 \leq r - \delta, \text{ for } i \in I, \text{ and } R^j_1 \geq r + \delta, \text{ for } j \notin I\}.$$

We provide a natural sufficient condition for $\mathbb{U}$ to be bounded.

Lemma 2 *Suppose that for some $\delta > 0$:*

$$\mu\left(\Omega^{\delta,I}\right) > 0, \tag{3.4}$$

for every subset $I \subset \{1,\dots,N\}$. Then $\mathbb{U}(x,y)$ is a bounded subset of $\mathbb{R}^+ \times \mathbb{R}^N$ for all $(x,y) \in \overline{\mathbb{L}}$.

Proof. It is clear that if $(c,\alpha) \in \mathbb{U}(x,y)$, then c must be bounded by above. Now suppose that there are $(c^m,\alpha^m) \in \mathbb{U}(x,y)$ so that $|\alpha^m|$ tends to infinity. Considering a subsequence, we may assume that all components of α^m converge (including the limit points $\pm\infty$). First assume that $(\alpha^m)^i$ converges to plus infinity for some i. Set I to be set of indices for which the limit point is plus infinity. Then, one can argue that on the set $\Omega^{\delta,I}$,

$$L\left((x-\alpha\cdot\vec{1}-g(\alpha)-c)(1+r),(y+\alpha)(1+R_1)\right)$$

converges to minus infinity. Hence a contradiction to the fact that $(c^m,\alpha^m) \in \mathbb{U}(x,y)$ and thus the above expression is non-negative with probability one.

Now, if $(\alpha^m)^i$ converges to minus infinity for some i. We set I to be the complement of the set on which the limit point is minus infinity and argue similarly.

□

4 Investment-consumption problem

In this model, we consider the classical problem of optimal investment and consumption of Merton [12, 9]. In our context of an infinite horizon, we assume that the investor derives utility from consumption. For a given initial position (x,y) and an admissible process $\nu = (c,\alpha) \in \mathbb{A}(x,y)$, the utility is given by:

$$\mathbb{J}(x,y,c,\alpha) := \mathbb{E}\left[\sum_{k=0}^{\infty}\rho^k U(c_k)\right], \tag{4.1}$$

where $U : \mathbb{R}^+ \to \mathbb{R}$, is a classical *utility function*, i.e., a concave, non-decreasing function satisfying the *Inada condition* and the given constant $\rho \in (0,1)$ is the *impatience parameter*. Then, the problem is to maximize the total expected utility function $\mathbb{J}$ over all admissible controls.

Remark 1 *We recall that in the finite horizon case, the utility considered in [1] for a given initial position (x,y), an horizon t and an admissible process $\nu = (c,\alpha) \in \mathbb{A}(x,y)$ is the following:*

$$\mathbb{J}(x,y,c,\alpha) := \mathbb{E}\left[\sum_{k=0}^{t-1}\rho^k U(c_k) + \rho^t \hat{U}(L(x_t,y_t))\right],$$

where $\hat{U}$ is as U. It is important to notice that when t is large, the second member of this utility function goes to 0. Indeed, for an admissible control (c,α), x_t and y_t are controllable, i.e. $(x_t,y_t) \subset A$ with A compact set, then $L(x_t,y_t) \subset L(A)$. Since $\hat{U}$ is an increasing function, $\hat{U}(L(x_t,y_t)) \subset U(\hat{A})$, hence for $\rho \in (0,1)$, $\lim_{t\to\infty}\rho^t\hat{U}(L(x_t,y_t)) = 0$.

Lemma 3 *For any admissible control (c,α), the utility function $\mathbb{J}$ is well defined and continuous.*

Proof. It is clear that if $(c,\alpha) \in \mathbb{U}(x,y)$, then c must be bounded by above, i.e., there is $M \in \mathbb{R}^+$ such that $c \leq M$. Then the function $\mathbb{J}$ is well defined since:

$$\sum_{k=0}^{\infty} \rho^k U(c_k) \leq \frac{U(M)}{1-\rho} \qquad \mathbb{P}-a.s.$$

For the continuity of $\mathbb{J}$, suppose that a sequence (x^n, y^n) converges to (x,y) $\mathbb{P}$-almost surely, then:

$$|\mathbb{J}(x^n,y^n) - \mathbb{J}(x,y)| \leq \mathbb{E}[\sum_{k=0}^{+\infty} \rho^k |U(c_k) - U(c_k^n)|]$$

Let $\varepsilon > 0$, be given. There exists T such that for every n

$$0 \leq \mathbb{E}[\sum_{k=T+1}^{+\infty} \rho^k U(c_k^n)] \leq \frac{\varepsilon}{3} \qquad \mathbb{P}-a.s.$$

and

$$0 \leq \mathbb{E}[\sum_{k=T+1}^{+\infty} \rho^k U(c_k)] \leq \frac{\varepsilon}{3} \qquad \mathbb{P}-a.s.$$

Since, we have $\forall n,\ \forall k,\ 0 \leq c_k \leq M,\ 0 \leq c_k^n \leq M$ $\mathbb{P}-a.s.$
Then,

$$|\mathbb{J}(x^n,y^n) - \mathbb{J}(x,y)| \leq \mathbb{E}[\sum_{k=0}^{T} \rho^k |U(c_k) - U(c_k^n)|] + \frac{2}{3}\varepsilon \quad \mathbb{P}-a.s.$$

Since, the function U is continuous, there exists an integer N such that, for every $n \geq N$, the first term of the second member of the previous inequality is less than $\frac{\varepsilon}{3}$ $\mathbb{P}$-almost surely. We have proved that $\mathbb{J}$ is continuous.

□

In what follows, the resulting optimal value is called the *value function* and is given by:

$$v(x,y) = \sup_{(c,\alpha)\in\mathbb{A}(x,y)} \mathbb{J}(x,y,c,\alpha).$$

However, it is well known that even with restrictions $(c,\alpha) \in \mathbb{A}(x,y)$, the value function may become infinite. The reason emanates from the market itself and is due to the possibility of arbitrage in the market. In this paper, we simply assume that

$$V(x,y) < \infty, \qquad \forall (x,y) \in \overline{\mathbb{L}}, \tag{4.2}$$

recall that $\mathbb{L}$ is defined in (3.1).

The value function has the following simple but important property:

Proposition 1 *The value function $V(\cdot,\cdot)$ is jointly concave and continuous on $\bar{\mathbb{L}}$.*

Proof. Note firstly that following the previous lemma, the set of admissible controls $\mathbb{A}(x,y)$ (resp. $\mathbb{U}(x,y)$ is convex. Now since U is concave, the concavity of the value function follows immediately and then, we conclude the continuity of $V(\cdot,\cdot)$ on $\overline{\mathbb{L}}$.

□

We continue by proving the dynamic programming. Recall that the set of admissible controls is given in (3.3).

Proposition 2 (Dynamic Programming)
Assume that (3.4) *is satisfied. Then for every* $(x,y) \in \overline{\mathbb{L}}$*, the value function satisfies the following equation:*

$$v(x,y) = \sup_{(c,\alpha)\in\mathbb{U}(x,y)} \mathbb{E}\left[U(c) + \rho\ v(x_1,y_1)\right], \tag{4.3}$$

where

$$(x_1,y_1) = \Big((x - \alpha\cdot\vec{1} - g(\alpha) - c)(1+r), (y+\alpha)(1+R_1)\Big).$$

Proof. or every $\varepsilon > 0$, there is $(c,\alpha) \in \mathbb{A}(x,y)$ so that

$$v(x,y) - \varepsilon \leq \mathbb{J}(x,y,c,\alpha) = \mathbb{E}\left[U(c_0) + \rho\left(\sum_{k=1}^{\infty} \rho^{k-1} U(c_k)\right)\right]. \tag{4.4}$$

Since $\mathbb{F}_k = \sigma(R_1,\ldots,R_k)$, and (c,α) are $\mathbb{F}$-adapted, for each $k \geq 1$, there are Borel measurable functions

$$(C_k, A_k) : (\mathbb{R}^N)^k \to \mathbb{R}^+ \times \mathbb{R}^N,$$

so that

$$(c_k,\alpha_k)(\omega) = (C_k,A_k)(\omega_1,\ldots,\omega_k).$$

For fix $\tilde{\omega} \in \mathbb{R}^N$ and define

$$(\tilde{c}_k,\tilde{\alpha}_k)(\omega) := (C_{k+1},A_{k+1})(\tilde{\omega},\omega_1,\ldots,\omega_k), \quad k \geq 0.$$

Then, it is clear that

$$(\tilde{c},\tilde{\alpha}) \in \mathbb{A}(x_1(\tilde{\omega}), y_1(\tilde{\omega})), \tag{4.5}$$

where

$$(x_1,y_1)(\tilde{\omega}) = (x_1 - \alpha_0\cdot\vec{1} - g(\alpha_0) - c_0)(1+r), (y_1+\alpha_0)(1+R_1(\tilde{\omega}))).$$

Then, we directly verify that

$$\mathbb{E}\left(\sum_{k=1}^{\infty} \rho^{k-1} U(c_k) \middle| \mathbb{F}_1\right)(\tilde{\omega}) = \mathbb{J}(x_1(\tilde{\omega}), y_1(\tilde{\omega}), \tilde{c}, \tilde{\alpha}).$$

Note that the right hand side of the above also depends on the initial controls (c_0,α_0). Also, the controls $(\tilde{c},\tilde{\alpha})$ depends on $\tilde{\omega}$. But we suppressed this dependences for notational simplicity.
By (4.4), we now have the following.

$$\begin{aligned} v(x,y) - \varepsilon &\leq \mathbb{E}\left[U(c_0) + \rho\mathbb{E}\left(\sum_{k=1}^{\infty} \rho^{k-1} U(c_k)\right) \middle| \mathbb{F}_1\right] \\ &= \mathbb{E}\left[U(c_0) + \rho\mathbb{J}(x_1(\tilde{\omega}), y_1(\tilde{\omega}), \tilde{c}, \tilde{\alpha})\right] \\ &= U(c_0) + \int_{\mathbb{R}^N} \mathbb{J}(x_1(\tilde{\omega}), y_1(\tilde{\omega}), \tilde{c}, \tilde{\alpha})\ \mu(d\tilde{\omega}) \\ &\leq \mathbb{E}\left[U(c_0) + \rho v(x_1,y_1)\right], \end{aligned}$$

where in the final inequality we used (4.5). Since $\varepsilon > 0$ is arbitrary and $(c_0,\alpha_0) \in \mathbb{U}(x,y)$, this proves one of inequality ($\leq$) in (4.3).

To prove the converse, we will use the fact that for each $(x,y) \in \overline{\mathbb{L}}, \mathbb{U}(x,y)$ is a bounded subset of $\mathbb{R}^+ \times \mathbb{R}^N$. Then, the existence of the value function arises from the fact that the initial optimization problem turns out to be a maximization problem of a continuous function on a compact set for the product topology. Since U and v are concave, hence continuous and $\mathbb{U}$ is compact under the above assumption, we conclude that there exists $(c^*, \alpha^*) \in \mathbb{U}(x,y)$ such that for every $(x_1^*, y_1^*) \in \overline{\mathbb{L}}$,

$$\mathbb{E}[U(c^*) + \rho v(x_1^*, y_1^*)] \geq \sup_{(c,\alpha)\in\mathbb{U}(x,y)} \mathbb{E}[U(c) + \rho v(x_1, y_1)] - \varepsilon$$

We then directly argue that

$$\begin{aligned} J(x,y,c^*,\alpha^*) &= \mathbb{E}[U(c^*) + \rho \sum_{k=1}^{\infty} \rho^{k-1} U(c_k^*)] \\ &\geq \mathbb{E}[U(c^*) + \rho J(x_1^*, y_1^*, c^*, \alpha^*)] - \varepsilon \\ &\geq \mathbb{E}[U(c^*) + v(x_1^*, y_1^*)] - 2\varepsilon. \end{aligned}$$

We now use the choice of $(c^*, \alpha^*) \in \mathbb{U}(x,y)$ and the inequality of $v \geq J$ to arrive at

$$\begin{aligned} v(x,y) &\geq J(x,y,c^*,\alpha^*) \geq \mathbb{E}[U(c^*) + \rho v(x_1^*, y_1^*)] - 2\varepsilon \\ &\geq \sup_{(c,\alpha)\in\mathbb{U}(x,y)} \mathbb{E}[U(c) + \rho v(x_1, y_1)] - 3\varepsilon. \end{aligned}$$

□

Now we can prove the existence and uniqueness of the value function:

Theorem 1 *Assume that* (3.4) *is satisfied, then the value function is the unique continuous solution of equation* (4.3)

Proof. First of all, let us begin by showing that V is a fixed point of a contraction mapping in the arbitrary Banach space of bounded functions. For this purpose, we denote by $B(\mathbb{U}(x,y))$ the space of all bounded functions on $\mathbb{U}(x,y)$. This space will be endowed with the sup-norm $\|h\| = \sup_{(c,\alpha)\in\mathbb{U}(x,y)} |h(x,y,c,\alpha)|$, where $h \in B(\mathbb{U}(x,y))$ and $(x,y) \in \overline{\mathbb{L}}$. We denote by T the operator defined for all $h \in B(\mathbb{U}(x,y))$ and all $(x,y) \in \overline{\mathbb{L}}$ by:

$$Th(x,y) = \sup_{(c,\alpha)\in\mathbb{U}(x,y)} \mathbb{E}[U(c) + \rho h(x_1, y_1)].$$

Let us check that T maps $(B(\mathbb{U}(x,y))$ into $B(\mathbb{U}(x,y))$:

$$\begin{aligned} \|Th(x,y)\| &= \| \sup_{(c,\alpha)\in\mathbb{U}(x,y)} \mathbb{E}[U(c) + \rho h(x_1, y_1)]\| \\ &\leq U(c) + \rho \sup_{(c,\alpha)\in\mathbb{U}(x,y)} \mathbb{E}[h(x_1, y_1)] \\ &\leq U(c) + \rho \|h\|. \end{aligned}$$

We claim that T is a ρ-contraction mapping. For this, we can see that T satisfied the Blackwell theorem (see the book of Le Van and Dana [10]):

- If $h \leq g$, then $\mathbb{U}(c) + \rho h(x_1, y_1) \leq \mathbb{U}(c) + \rho h(x_1, y_1)$ and therefore $Th \leq Tg$.
- There exists $\alpha \in]0,1[$, take $\alpha = \rho$ such that $T(h+a) \leq Th + a\alpha$, for $a \in \mathbb{R}^+$.

By Banach fixed point theorem for contraction mappings, there exists a unique fixed point of T in $B(\mathbb{U}(x,y))$ which is $V(x,y)$.

□

Remark 2 *One of the well-known properties of contraction mappings is that the fixed point is the limit of the sequence $(T^n h)$ when n converges to infinity, where h is any element of $B(\mathbb{U}(x,y))$. More precisely, we have, $\lim_{n\to\infty} \|T^n h - v\| = 0$. In particular, we have:*
$\forall (x,y) \in \bar{\mathbb{L}}, V(x,y) = \lim_{n\to\infty} T^n h(x,y)$.

Remark 3 *Assume that the utility function U takes non-negative values. Then we can show that the value function of (4.3) satisfies the transversality condition:*

$$\forall (c,\alpha) \in \mathbb{U}(x,y), \lim_{T\to+\infty} \rho^T v(x_T, y_T) = 0.$$

Indeed, let (c,α) be in $\mathbb{A}(x,y)$, since $\mathbb{U}$ is a bounded subset of $\mathbb{R}^+ \times \mathbb{R}^N$ and the utility function U is non-negative, we have:

$$\sup_{(c,\alpha)\in\mathbb{A}(x,y)} \mathbb{J}(x,y,c,\alpha) \leq \frac{U(M)}{1-\rho}.$$

Hence,

$$\forall T, \qquad \lim_{T\to\infty} \rho^T v(x_T, y_T) = 0.$$

Remark 4 *By using the property obtained in Remark 3, we can show the uniqueness of the value function differently. Indeed, let V be a continuous solution of the dynamic programming equation* (4.3)*, we have:*

$$\forall T; \quad V(x,y) = U(c_0) + \rho U(c_1) + \rho^2 U(c_2) + \cdots + \rho^{T-1} U(c_{T-1}) + \rho^T V(x_T, y_T).$$

Since $\rho^T V(x_T, y_T) \to 0$, hence

$$V(x,y) = \sum_{k=0}^{+\infty} \rho^k U(c_k) \leq v(x,y).$$

To prove the opposite inequality, let (c,α) be any admissible control in $\mathbb{A}(x,y)$, then

$$(c_k, \alpha_k) \in \mathbb{U}(x_k, y_k), \qquad \forall k \geq 1,$$

We use the construction given in Proposition 2 to conclude that:

$$V(x_k, y_k) \geq \mathbb{E}\left[U(c_k) + \rho V(x_{k+1}, y_{k+1}) \mid \mathbb{F}_k\right], \qquad k = 0, 1, \ldots, T-1.$$

We iterate this inequality, we obtain

$$\begin{aligned} V(x,y) &\geq \mathbb{E}\left[U(c_0) + \rho V(x_1,y_1)\right] \\ &\geq \mathbb{E}\left[U(c_0) + \rho \mathbb{E}\left[U(c_1) + \rho V(x_2,y_2) \mid \mathbb{F}_1\right]\right] \\ &= \mathbb{E}\left[U(c_0) + \rho U(c_1) + \rho^2 V(x_2,y_2)\right] \\ &\geq \mathbb{E}\left[\sum_{k=0}^{T-1} \rho^k U(c_k) + \rho^T V(x_T,y_T)\right] \end{aligned}$$

Since $\lim_{T\to\infty} \rho^T v(x_T,y_T) = 0$ *and* $(c,\alpha) \in \mathbb{A}(x,y)$ *is arbitrary, the above implies*

$$V(x,y) \geq \sup_{(c,\alpha)\in\mathbb{A}(x,y)} \mathbb{J}(x,y,c,\alpha) = v(x,y).$$

References

[1] S. Chebbi and H.M. Soner, Merton problem in a discrete market with frictions, Nonlinear Analysis: Real World Applications 14 (2013) 179-187.

[2] G.M. Constantinides, Capital market equilibrium with transaction costs, Journal of Political Economy 94 (1986) 842-862.

[3] U. Çetin, R. Jarrow and P. Protter, Liquidity risk and arbitrage pricing theory, Finance and Stochastics 8 (2004) 311-341.

[4] U. Çetin and L.C.G. Rogers, Modelling liquidity effects in discrete time, Mathematical Finance 17 (2007) 15-29

[5] M.H.A. Davis and A.R. Norman, Portfolio selection with transaction costs, Mathematics of Operations Research 15(4) (1990) 676-713.

[6] B. Dumas and E. Luciano, An exact solution to a dynamic portfolio choice problem under transaction costs, Journal of Finance 46 (1991) 577-595.

[7] S. Goekey and H.M. Soner, Liquidity in a binomial market, Mathematical Finance 22 (2012) 250-276.

[8] E. Jouini and E. Kallal, Martingales and arbitrage in securities markets with transaction costs, Journal of Economic Theory 66 (1995) 178-197.

[9] I. Karatzas and S.E. Shreve, Methods of Mathematical Finance, Springer-Verlag, 1998.

[10] C. Le Van and Rose-Anne Dana, Dynamic programming in economics, Kluer Academic Publishers, 2003.

[11] M.J.P. Magill and G.M. Constantinides, Portfolio selection with transaction costs, Journal of Economic Theory 13 (1976) 254-263.

[12] R.C. Merton, Optimum consumption and portfolio rules in a continuous time case, Journal of Economic Theory 3 (1971) 373-413.

[13] S.E. Shreve and H.M. Soner, Optimal investment and consumption with transaction costs, The annals of Applied Probability 4(3) (1994) 609-692.