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Merton problem in a discrete market with frictions[☆]

Souhail Chebbi [a,*], Halil Mete Soner [b,c]

[a] *Department of Mathematics, King Saud University, Riyadh, Saudi Arabia*
[b] *Department of Mathematics, Swiss Federal Institute of Technology (ETH), Zurich, Switzerland*
[c] *Swiss Finance Institute, Zurich, Switzerland*

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ABSTRACT

We study the classical optimal investment and consumption problem of Merton in a discrete time model with frictions. Market friction causes the investor to lose wealth due to trading. This loss is modeled through a nonlinear penalty function of the portfolio adjustment. The classical transaction cost and the liquidity models are included in this abstract formulation. The investor maximizes her utility derived from consumption and the final portfolio position. The utility is modeled as the expected value of the discounted sum of the utilities from each step. At the final time, the stock positions are liquidated and a utility is obtained from the resulting cash value. The controls are the investment and the consumption decisions at each time. The utility function is maximized over all controls that keep the after liquidation value of the portfolio non-negative. A dynamic programming principle is proved and the value function is characterized as its unique solution with appropriate initial data. Optimal investment and consumption strategies are constructed as well.

1. Introduction

In his seminal paper [1], Merton formulated and solved a central financial problem of optimal investment and consumption in continuous time. This problem has been extended and extensively used in financial models. Instead of surveying this large body of scientific accomplishments, we simply refer the reader to the book of Karatzas and Shreve [2], the classical paper of Sethi and Taksar [3] and the recent papers [4–10] and the references therein. One direction of extension has been to include market frictions in the Merton problem and to study their impact on the optimal decisions. This paper is a study in this direction.

There are several sources of market friction. However, in the literature transaction costs and portfolio constraints have received the most attention. The proportional transaction cost is a consequence of the bid and ask spread and it was first studied in the context of the Merton problem by Magill and Constantinides [11] and later by Constantinides [12]. This model was later put in a modern mathematical framework in continuous time by Davis and Norman [13], the maximal growth rate problem was solved explicitly by Dumas and Luciano [14] and the viscosity theory was developed by Shreve and Soner [15]. In discrete theory, the model is fully developed by Jouini and Kallal [16]. For the results on transaction costs, we refer the reader to the recent book by Kabanov and Safarian [17] and the references therein. A similar concept of friction is due to liquidity. Indeed, recently Cetin, Jarrow and Protter developed a mathematical model for an illiquid market in [18]. The main modeling concept in [18] is the notion of a supply curve which gives the price of stock as *function of the trade size*.

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* Corresponding author.
E-mail addresses: schebbi@ksu.edu.sa (S. Chebbi), hmsoner@ethz.ch (H.M. Soner).

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Numerical studies for the existence of a supply curve were then provided in [19]. In continuous time, only the problem of super-replication of a contingent claim was recently studied by Cetin et al. [20]. Later, this analysis was carried out in discrete time by Gokay and Soner [21] and probabilistically by Dolinsky and Soner [22]. Also the Merton problem with this model was analyzed by Cetin and Rogers [23]. They showed that the marginal price process is a martingale under the usual risk neutral measure.

Our approach is similar to that of [23] and we provide several extensions by studying the problem in multi-dimensions, in a generality that covers both the transaction cost and illiquidity. In addition, we also study the full optimal investment and the *consumption* problem.

The model we consider postulates a penalty function for trading. To simplify the discussion, let us assume in this Introduction, that there is only one risky asset (in this paper, we consider the general multi-dimensional case). If at any time the investor decides to make a portfolio adjustment of α dollars in her stock account, then she loses $g(\alpha)$ dollars to market friction. This function is assumed to be a general non-negative convex function with $g(0) = 0$. In the case of proportional transaction costs $g(\alpha) = \lambda|\alpha|$ and in one particular example of an illiquid market with no bid and ask spread $g(\alpha) = \lambda\alpha^2$, where $\lambda > 0$ is a (small) market parameter. The discrete time formulation has the advantage of studying several different types of market frictions together through a general penalty function g. Indeed, in continuous time only the structure of g near the origin is relevant. Hence, one has to distinguish the cases in which g differentiable at the origin and when not. The corresponding techniques, results and the equations crucially depend on this differentiability property. In contrast to continuous time, a unifying approach is possible in discrete time.

In Section 2, we define this market. In particular, the notion of a solvency set is given in Section 2.2. This is similar to the notion used in the transaction cost literature [13]. The dynamics of the cash and stock positions are postulated in Eqs. (2.2) and (2.3), respectively. We then formulate the optimal investment and the consumption problem in Section 3. The problem we choose to study is the finite horizon problem. The investor's utility is derived from the sum of all discounted utilities from consumption plus the utility from the final liquidation value of the portfolio. The dynamic programming is introduced and proved in Section 4 in Proposition 1. We then utilize it to characterize the value function as the unique solution of the dynamic programming equation (4.1) together with the natural initial condition. The optimal strategies are also constructed in Section 5. This main result, Theorem 1 is also stated and proved in that section.

2. Model

We consider a market with a money market account and N risky assets. We assume that the money market account pays a return of fraction $r > 0$ of the invested amount. The problem has a finite horizon T which is a given positive integer. The risky assets, called the stocks, provide a random return of $(R_k)_{k=1}^T$ with values in $[-1, \infty)^N$. To achieve simplicity, we assume that the returns are identically and independently distributed over time. We let μ be the common probability measure of R_k's. Mathematically, we consider the probability space $(\Omega, \mathbb{F}, \mathbb{P})$, where

$$\Omega = (\mathbb{R}^N)^T,$$

for $k \in \{1, \ldots, T\}$, $B_k(\omega) = \omega_k$ is the canonical map, $\mathbb{F} = (\mathrm{F}_k)_{k=1}^T$ is the standard filtration generated by the canonical map, i.e. for $k \in \{1, \ldots, T\}$,

$$\mathrm{F}_k = \sigma(B_1, \ldots, B_k),$$

F_0 is the trivial filtration and $\mathbb{P}$ is the product probability measure given by

$$\mathbb{P}\left(\{\omega \in \Omega : \omega_k \in A_k,\ k = 1, \ldots, T\}\right) = \prod_{k=1}^{T} \mu(A_k).$$

The existence of such a probability space is standard. We then let the return vector at time k be given by

$$R_k(\omega) := B_k(\omega) = \omega_k.$$

Then, R_k's are F_k measurable, hence R is an $\mathbb{R}^N$-valued, $\mathbb{F}$-adapted process. The connection between the stock prices S_k^i of the ith stock at time $k \in \{1, \ldots, T\}$ and the return process R is simply given by

$$S_k^i = S_0^i \prod_{j=1}^{k} [1 + R_j^i] \iff R_k^i = \frac{S_k^i - S_{k-1}^i}{S_{k-1}^i}, \quad i = 1, \ldots, N,$$

where S_0^i is the initial stock value. Since $R_j^i \geq -1$, S is an $(\mathbb{R}^+)^N$ valued, $\mathbb{F}$-adapted process.

The portfolio position of the investor is an $\mathbb{F}$-adapted, $\mathbb{R} \times (\mathbb{R}^+)^N$-valued process (x, y) and it has the following interpretation,

$$x_k = \text{money invested in the money market account at time } k,$$

$$y_k^i = \text{money invested in the } i\text{-th stock at time } k \textit{ prior} \text{ to the portfolio adjustment}.$$

For $k \in \{1, \ldots, T\}$, let z_k be the number of stocks held at time k *prior* to the portfolio adjustment. Hence, z_k is F_{k-1} measurable and z is an $\mathbb{F}$-*predictable* process with values in $\mathbb{R}^N$. Moreover,

$$y_k^i = z_k^i S_k^i, \quad i = 1, \ldots, N, \ \ k = 0, 1, \ldots, T.$$

2.1. Dynamics

Without market friction, self-financing condition would imply that a portfolio adjustment of $\Delta_k z := z_{k+1} - z_k$ at time $k \in \{1, \ldots, T\}$, requires a change of $S_k \cdot \Delta_k z$ dollars in the cash account. However, in markets with friction we know that trading results in a loss which is a certain small percentage of the traded dollar amount

$$\alpha_k^i := S_k^i \Delta_k z^i = S_k^i \, (z_{k+1}^i - z_k^i), \quad i = 1, \ldots, N, \;\; k = 1, \ldots, T. \tag{2.1}$$

We thus postulate the following dynamics for the cash position,

$$x_{k+1} = \left(x_k - \alpha_k \cdot \vec{1} - g(\alpha_k) - c_k\right)(1 + r), \qquad k \in \{1, \ldots, T\}, \tag{2.2}$$

where the non-negative, $\mathbb{F}$-adapted process c is the *consumption* of the investor, $\vec{1} := (1, \ldots, 1)$ and hence for any vector $\eta \in \mathbb{R}^N$

$$\eta \cdot \vec{1} = \sum_{i=1}^{N} \eta_i,$$

and

$$g : \mathbb{R}^N \to [0, \infty),$$

is the given *loss function*. We assume that

$$g(0) = 0, \quad g \geq 0, \; g \text{ is convex}.$$

Specific examples of a loss function are

$$g(\alpha) = \sum_{i=1}^{N} \lambda_i |\alpha^i|, \quad \text{or} \quad g(\alpha) = \sum_{i=1}^{N} \lambda_i (\alpha^i)^2,$$

where λ^i's are given non-negative (small) constants. The first of the above example corresponds to the classical example of the proportional costs [13,14,16,11,15]. The second, however, is a model of illiquidity as formulated by Cetin et al. [18] and in discrete time in [23,21]. The main difference between the two examples is the differentiability at the origin. Indeed, a non-differentiability of g at the origin corresponds to a proportional transaction cost, or equivalently the existence of a bid–ask spread in the market.

The dynamics of the y process is the classical one. Indeed, for $k \in \{0, 1, \ldots, T-1\}$,

$$\begin{aligned}
y_{k+1}^i &= y_k^i + \left[z_{k+1}^i S_{k+1}^i - z_k^i S_k^i\right] \\
&= y_k^i + S_k^i \left[z_{k+1}^i - z_k^i\right] + z_{k+1}^i \left[S_{k+1}^i - S_k^i\right] \\
&= y_k^i + \alpha_k^i + S_k^i z_{k+1}^i \left(\frac{S_{k+1}^i - S_k^i}{S_k^i}\right) \\
&= y_k^i + \alpha_k^i + \left[S_k^i (z_{k+1}^i - z_k^i) + z_k^i S_k^i\right] \, R_{k+1}^i \\
&= y_k^i + \alpha_k^i + \left(\alpha_k^i + y_k^i\right) \, R_{k+1}^i \\
&= \left(y_k^i + \alpha_k^i\right) \, (1 + R_{k+1}^i).
\end{aligned} \tag{2.3}$$

Notice that the dynamics of the state variables (x, y) in (2.2)–(2.3) are given only through the process α and not z. Hence, in whatever follows, we use the $\mathbb{F}$-adapted process α instead of z.

We also note that the mark-to-market value

$$w_k := x_k + y_k \cdot \vec{1} = x_k + \sum_{i=1}^{N} y_k^i$$

satisfies the equation

$$\begin{aligned}
w_{k+1} &= w_k + r x_k + [\alpha_k + y_k] \cdot R_{k+1} - \alpha_k \cdot \vec{1} r - c_k(1 + r) - g(\alpha_k)(1 + r) \\
&= w_k \left[1 + r + \pi_k \cdot (R_{k+1} - r)\right] - c_k(1 + r) - g(\alpha_k)(1 + r),
\end{aligned}$$

where $\pi_k^i := [\alpha_k^i + y_k^i]/w_k$ is the fraction of the mark-to-market value invested in the stock after the portfolio adjustment. Indeed, this is the classical wealth equation when there is no friction, i.e., when $g \equiv 0$.

Remark 1. One may also use a model in which the loss function

$$g\,(\alpha_k) = g\left(S_k^i [z_{k+1}^i - z_k^i]\right),$$

is replaced by

$$G\left([z^i_{k+1} - z^i_k]\right) S^i_k,$$

with some convex function G. Although such a model has different scaling properties, same results can be proved with the methods developed here. □

2.2. Solvency region

It is well known that any optimization problem in these models requires a lower bound on the wealth like variables; see [2]. Otherwise, one may easily obtain non-intuitive trivial results as consumption with no bound would be admissible. In this context, an appropriate notion is to require the after-liquidation value of the portfolio to be non-negative.

For portfolio position $(x, y) \in \mathbb{R} \times \mathbb{R}^N$, the *after-liquidation value* is defined simply as the cash value of the position after the investor is forced to liquidate (i.e., sell or close) all stock positions. Due to the loss function postulated in (2.2) this value differs from the mark-to-market value defined in the previous subsection. Indeed, using the idea behind (2.2), with $z_k = y_k/S_k$, $z_{k+1} = 0$, we obtain $\alpha_k = -y_k$ and

$$L(x, y) := x + y \cdot \vec{1} - g(-y), \quad \text{and} \quad \mathbb{L} := \left\{(x, y) \in \mathbb{R} \times \mathbb{R}^N : L(x, y) > 0\right\}. \tag{2.4}$$

Then, the solvency condition is simply given by the requirement that $L(x_k, y_k) \geq 0$ for all $k \in \{1, \ldots, T\}$, $\mathbb{P}$-almost surely. Using this requirement we define the set of admissible controls as follows.

Definition 1. A *control process* $\nu := (c, \alpha)$ consists of a non-negative, $\mathbb{F}$-adapted consumption process c and an $\mathbb{R}^N$-valued, $\mathbb{F}$-adapted portfolio adjustment process α. We say that a control process $\nu = (c, \alpha)$ is *admissible* with initial position $(x, y) \in \bar{\mathbb{L}}$, if the solution $(x_k, y_k)_{k=0}^T$ corresponding to (2.2)–(2.3) with initial data $x_0 = x$, $y_0 = y$ and controls (c, α) satisfies

$$L(x_k, y_k) = x_k + y_k \cdot \vec{1} - g(-y_k) \geq 0, \iff (x_k, y_k) \in \bar{\mathbb{L}}, \quad \forall\, k \in \{1, \ldots, T\},$$

$\mathbb{P}$-almost surely. We denote by $\mathbb{A}(x, y)$ the set of all *admissible controls*. □

Notice that starting from a position $(x, y) \in \bar{\mathbb{L}}$, the initial choice of (c_0, α_0) an investor can make is also restricted by the above definition. Indeed, she can only choose pairs (c_0, α_0) that satisfy

$$L(x_1, y_1) = L\Big((x - \alpha_0 \cdot \vec{1} - g(\alpha) - c_0)(1 + r), (y + \alpha_0)(1 + R_1)\Big) \geq 0,$$

$\mathbb{P}$-almost surely. The above is a nonlinear restriction which can be solved in specific examples. In the general context, we simply define

$$\mathbb{U}(x, y) := \left\{(c, \alpha) \in \mathbb{R}^+ \times \mathbb{R}^N : L(x_1, y_1) \geq 0,\ \mathbb{P}\text{-a.s.}\right\}. \tag{2.5}$$

We may rewrite the admissibility criterion using the sets $\mathbb{U}(x, y)$ as well. For future reference, we record this simple connection,

$$(c, \alpha) \in \mathbb{A}(x, y) \iff (c_k, \alpha_k) \in \mathbb{U}(x_k, y_k), \quad \forall\, k = 0, 1, \ldots, T - 1, \tag{2.6}$$

where (x_k, y_k) is the solution of (2.2)–(2.3).

Notice that for $(x, y) \in \bar{\mathbb{L}}$, by definition, the choice $c = 0$ and $\alpha = -y$ provides an admissible pair. Indeed, we have the following lemma.

Lemma 1. *For any $(x, y) \in \bar{\mathbb{L}}$, we have*

$$\mathbb{U}(x, y) \neq \emptyset, \qquad \mathbb{A}(x, y) \neq \emptyset.$$

Proof. Indeed, the choices $c_0 = 0$ and $\alpha_0 = -y$ imply that

$$\begin{aligned} L\Big((x - \alpha_0 \cdot \vec{1} - g(\alpha_0) - c_0)(1 + r), (y + \alpha_0)(1 + R_1)\Big) &= L\Big((x + y \cdot \vec{1} - g(-y))(1 + r), 0\Big) \\ &= (x + y \cdot \vec{1} - g(-y))(1 + r) \geq 0, \end{aligned}$$

since $(x, y) \in \bar{\mathbb{L}}$ is equivalent to $x + y \cdot \vec{1} - g(-y) \geq 0$. Hence $(0, -y) \in \mathbb{U}(x, y)$. To construct an admissible process, set $c \equiv 0$, $\alpha_0 = -y$ and $\alpha_k = 0$ for all $k \geq 1$. Then, the solution of (2.2)–(2.3) at time $k \in \{1, \ldots, T\}$ is given by $y_k = 0$ and

$$x_k = (x + y \cdot \vec{1} - g(-y))(1 + r)^k.$$

Then,

$$L(x_k, y_k) = x_k = (x + y \cdot \vec{1} - g(-y))(1 + r)^k \geq 0. \quad \square$$

Notice that if the support of the measure μ is unbounded from above, there might be an implicit short-sell constraint. Indeed, in that case any short position may yield a possible unbounded loss with non-zero probability.

We give the example of a one-dimensional Binomial model to clarify some of the above notations.

Example 1. Suppose there is only one risky asset with a binomial return structure. Then, there are $0 < d < u$ such that the return $R = u$ with probability $p \in (0, 1)$ and $R = d$ with probability $(1 - p)$. Although in a market with friction parameters d, r, u can be arbitrary, let us assume that $d < r < u$. Then, when $\alpha \geq 0$, the worst case for the restriction occurs after a down movement of the stock and places the constraint

$$L((x - \alpha - g(\alpha) - c)(1 + r), (y + \alpha)(1 + d)) \geq 0.$$

Set $D := (1 + d)/(1 + r)$ so that the above inequality is equivalent to

$$0 \leq c \leq x + yD - \alpha(1 - D) - g(\alpha) - g(-(y + \alpha)(1 + d))/(1 + r).$$

When $\alpha \leq 0$, the worst case occurs after an up movement and a similar calculation with $G := (1 + u)/(1 + r)$ yields,

$$0 \leq c \leq x + yG + \alpha(G - 1) - g(\alpha) - g(-(y + \alpha)(1 + u))/(1 + r).$$

Combining, we obtain

$$\begin{aligned}\mathbb{U}(x, y) = &\{(c, \alpha) \in (\mathbb{R}^+)^2 : c \leq x + yD - \alpha(1 - D) - g(\alpha) - g(-(y + \alpha)(1 + d))/(1 + r)\}\\ &\cup \{(c, \alpha) \in \mathbb{R}^+ \times \mathbb{R}^- : c \leq x + yG + \alpha(G - 1) - g(\alpha) - g(-(y + \alpha)(1 + u))/(1 + r)\}. \quad \Box\end{aligned}$$

3. Investment–consumption problem

In this model, we consider the classical problem of optimal investment and consumption of Merton [1,2]. We assume that the investor derives utility from consumption and the final portfolio position. For a given initial position (x, y), a horizon $t \in \{1, \ldots, T\}$ and an admissible process $\nu = (c, \alpha) \in \mathbb{A}(x, y)$, the utility is given by

$$\mathbb{J}(t, x, y, c, \alpha) := \mathbb{E}\left[\sum_{k=0}^{t-1} \rho^k U(c_k) + \rho^t \widehat{U}\left(L\left(x_t, y_t\right)\right)\right], \tag{3.1}$$

where

$$U, \widehat{U} : \mathbb{R}^+ \to \mathbb{R},$$

are classical *utility functions*, i.e., a concave, non-decreasing functions satisfying the Inada conditions (see [2]) and the given constant $\rho \in (0, 1)$ is the *impatience parameter*.

Then, the problem is to maximize $\mathbb{J}$ over all admissible controls. The resulting optimal value is called the *value function* and is given by

$$v(t, x, y) := \sup_{(c,\alpha)\in\mathbb{A}(x,y)} \mathbb{J}(t, x, y, c, \alpha). \tag{3.2}$$

However, it is well known that even with the restriction $(c, \alpha) \in \mathbb{A}(x, y)$ the value function may become infinite. The reason emanates from the market itself and is due to the possibility of arbitrage in the market. In markets without friction (i.e. when $g \equiv 0$) one has to assume the existence of a martingale measure to avoid arbitrage. However, when there is friction this condition may not be needed. This is demonstrated by Cetin and Rogers [23] in the illiquid markets. In this paper, we simply assume that

$$V(t, x, y) < \infty, \quad \forall t \in \{1, \ldots, T\},\ (x, y) \in \bar{\mathbb{L}}, \tag{3.3}$$

recall that $\mathbb{L}$ is defined in (2.4).

Remark 2. Consider a horizon $t \in \{1, \ldots, T\}$. Then, to define the maximization problem (3.1)–(3.2), we only need the controls (c_k, α_k) for $k = 0, 1, \ldots, t - 1$ and the initial condition (x_0, y_0). Now suppose that one constructs the controls satisfying the admissibility criterion (2.6) up to time $t - 1$. Then, one may extend this control to times larger and equal to k by simply setting the controls to be zero. This extension satisfies the constraint (2.6) and hence it is admissible. Further, this extension would not change the value of $\mathbb{J}$. $\Box$

4. Dynamic programming

In this section, we prove the dynamic programming. In fact, the reason of introducing the problem not only for T but also for every $t \in \{1, \ldots, T\}$ is due to this.

We start with a simple but an important property.

Lemma 2. *The admissible class of controls $\mathbb{A}(x, y)$ is a convex set and for every $t \in \{1, \ldots, T\}$ the value function $v(t, \cdot, \cdot)$ is jointly concave on $\bar{\mathbb{L}}$.*

Proof. For initial data $(x_0^i, y_0^i) \in \bar{\mathbb{L}}$ and controls $\nu^i := (c^i, \alpha^i) \in \mathbb{A}(x_0^i, y_0^i)$, $i = 1, 2$, let (x^i, y^i) be the solutions of (2.2)–(2.3) with initial data (x_0^i, y_0^i) and control ν^i. Set $\bar{\nu} := (\nu^1 + \nu^2)/2$ and let $(\bar{x}, \bar{y})$ be the solutions of (2.2)–(2.3) with initial data $(\bar{x}_0, \bar{y}_0)$ and control $\bar{\nu}$. Since g is assumed to be convex with $g(0) = 0$, we directly verify that

$$2g(\bar{\alpha}_k) \le g(\alpha_k^1) + g(\alpha_k^2).$$

Hence, (2.2) implies that

$$2\bar{x}_k \ge x_k^1 + x_k^2.$$

Also, it is clear from (2.3) that

$$2\bar{y}_k = y_k^1 + y_k^2.$$

We again use the convexity of g, $g(0) = 0$ and the definition of L to arrive at

$$2L(\bar{x}_k, \bar{y}_k) \ge L(x_k^1, y_k^1) + L(x_k^2, y_k^2) \ge 0.$$

Hence $\bar{\nu} \in \mathbb{A}(\bar{x}, \bar{y})$. Since U and $\widehat{U}$ are concave, the concavity of the value function follows immediately. □

We continue by proving the dynamic programming. Recall that the set of admissible controls is given in Definition 1 and the set $\mathbb{U}(x, y)$ in (2.5).

Proposition 1 (*Dynamic Programming*)**.** *Assume (3.3). Then for every $t \in \{1, \dots, T\}$ and $(x, y) \in \bar{\mathbb{L}}$*

$$v(t, x, y) = \sup_{(c,\alpha)\in\mathbb{U}(x,y)} \mathbb{E}\left[U(c) + \rho\, v\,(t-1, x_1, y_1)\right], \tag{4.1}$$

where

$$(x_1, y_1) = \Big((x - \alpha\cdot\vec{1} - g(\alpha) - c)(1+r),\, (y+\alpha)(1+R_1)\Big).$$

Moreover, v also satisfies the initial condition

$$v(0, x, y) = \widehat{U}\,(L(x, y)), \quad \forall (x, y) \in \bar{\mathbb{L}}. \tag{4.2}$$

Proof. For every $\epsilon > 0$ there is $(c, \alpha) \in \mathbb{A}(x, y)$ so that

$$v(t, x, y) - \epsilon \le \mathbb{J}(t, x, y, c, \alpha) = \mathbb{E}\left[U(c_0) + \rho\left(\sum_{k=1}^{t-1}\rho^{k-1}U(c_k) + \rho^{t-1}\widehat{U}\,(L(x_t, y_t))\right)\right]. \tag{4.3}$$

Since $\mathrm{F}_k = \sigma(R_1, \dots, R_k)$, and (c, α) are $\mathbb{F}$-adapted, for each $k \in \{1, \dots, T\}$, there are Borel measurable functions

$$(C_k, A_k) : (\mathbb{R}^N)^k \to \mathbb{R}^+ \times \mathbb{R}^N,$$

so that

$$(c_k, \alpha_k)(\omega) = (C_k, A_k)(\omega_1, \dots, \omega_k).$$

Fix $\tilde{\omega} \in \mathbb{R}^N$ and define

$$(\tilde{c}_k, \tilde{\alpha}_k)(\omega) := (C_{k+1}, A_{k+1})(\tilde{\omega}, \omega_1, \dots, \omega_k), \quad k = 0, 1, \dots, T-1.$$

This defines the control $\tilde{\nu}$ up to time $T-1$. As in Remark 2, we extend by setting it to zero at time T. Then, it is clear that

$$(\tilde{c}, \tilde{\alpha}) \in \mathbb{A}(x_1(\tilde{\omega}), y_1(\tilde{\omega})), \tag{4.4}$$

where

$$(x_1, y_1)(\tilde{\omega}) = \Big((x_1 - \alpha_0\cdot\vec{1} - g(\alpha_0) - c_0)(1+r),\, (y_1 + \alpha_0)(1 + R_1(\tilde{\omega}))\Big).$$

Then, we directly verify that

$$\mathbb{E}\left(\sum_{k=1}^{t-1}\rho^{k-1}U(c_k) + \rho^{t-1}\widehat{U}\,(L(x_t, y_t))\,\middle|\, \mathrm{F}_1\right)(\tilde{\omega}) = \mathbb{J}(t-1, x_1(\tilde{\omega}), y_1(\tilde{\omega}), \tilde{c}, \tilde{\alpha}).$$

Note that the right hand side of the above also depends on the initial controls (c_0, α_0). Also, the controls $(\tilde{c}, \tilde{\alpha})$ depend on $\tilde{\omega}$. But we suppressed these dependences for notational simplicity.

By (4.3), we now have the following.

$$\begin{aligned} v(t,x,y)-\epsilon &\leq \mathbb{E}\left[U(c_0)+\rho\mathbb{E}\left(\sum_{k=1}^{t-1}\rho^{k-1}U(c_k)+\rho^{t-1}\widehat{U}\left(L(x_t,y_t)\right)\middle| \mathrm{F}_1\right)\right] \\ &= \mathbb{E}\left[U(c_0)+\rho\mathbb{J}(t-1,x_1(\tilde\omega),y_1(\tilde\omega),\tilde c,\tilde\alpha)\right] \\ &= U(c_0)+\int_{\mathbb{R}^N}\mathbb{J}(t-1,x_1(\tilde\omega),y_1(\tilde\omega),\tilde c,\tilde\alpha)\,\mu(d\tilde\omega) \\ &\leq \mathbb{E}\left[U(c_0)+\rho v(t-1,x_1,y_1)\right], \end{aligned}$$

where in the final inequality we used (4.4). Since $\epsilon>0$ is arbitrary and $(c_0,\alpha_0)\in\mathbb{U}(x,y)$, this proves one of the inequality $(\leq)$ in (4.1).

The opposite inequality is proved by using the concavity and hence the continuity of the value function and a measurable selection argument. Indeed, for $\epsilon>0$ and $(x,y)\in\bar{\mathbb{L}}$ choose $(c^\epsilon(x,y),\alpha^\epsilon(x,y))\in\mathbb{A}(x,y)$ so that

$$J(t-1,x,y,c^\epsilon(x,y),\alpha^\epsilon(x,y))\geq v(t-1,x,y)-\epsilon, \tag{4.5}$$

also $(c^\epsilon(x,y),\alpha^\epsilon(x,y))\in\mathbb{A}(\tilde x,\tilde y)$ for all $(\tilde x,\tilde y)$ in a neighborhood of (x,y). Now define an open set $O^\epsilon(x,y)$ by $(\tilde x,\tilde y)\in O^\epsilon(x,y)$ if and only if the following conditions hold,

$$\begin{aligned} &(c^\epsilon(x,y),\alpha^\epsilon(x,y))\in\mathbb{A}(\tilde x,\tilde y), \\ &\left|J(t-1,x,y,c^\epsilon(x,y),\alpha^\epsilon(x,y))-J(t-1,\tilde x,\tilde y,c^\epsilon(x,y),\alpha^\epsilon(x,y))\right|<\epsilon, \end{aligned} \tag{4.6}$$

and

$$\left|v(t-1,x,y)-v(t-1,\tilde x,\tilde y)\right|<\epsilon. \tag{4.7}$$

We use the continuity of J and v to ensure that $O^\epsilon(x,y)$ is an open set of $\bar{\mathbb{L}}$ for each (x,y). Since $\bar{\mathbb{L}}$ is locally compact and $O^\epsilon(x,y)$ is an open covering, there exists a countable set $\{(x_k,y_k)\}$ so that,

$$\bar{\mathbb{L}}=\cup_k O^\epsilon(x_k,y_k).$$

By a standard argument we then construct a *disjoint* covering Γ_k and each is a subset of $O^\epsilon(x_k,y_k)$. Then, for every point $(\tilde x,\tilde y)\in\bar{\mathbb{L}}$ there exists a unique $k=k(\tilde x,\tilde y)$ so that $(\tilde x,\tilde y)\in\Gamma_k$.

We now choose $(c^*,\alpha^*)\in\mathbb{U}(x,y)$ so that

$$\mathbb{E}\left[U(c^*)+\rho\, v\left(t-1,x_1^*,y_1^*\right)\right]\geq\sup_{(c,\alpha)\in\mathbb{U}(x,y)}\mathbb{E}\left[U(c)+\rho\, v\,(t-1,x_1,y_1)\right]-\epsilon,$$

where (x_1^*,y_1^*) is the random point that is obtained from the initial data (x,y) and the control (c^*,α^*). We now define a control $(\hat c,\hat\alpha)\in\mathbb{A}(x,y)$ by

$$(\hat c_s,\hat\alpha_s)=\begin{cases}(c^*,\alpha^*), & \text{if } s=0,\\ (c^\epsilon(x_k,y_k)_{s-1},\alpha^\epsilon(x_k,y_k)_{s-1}) & \text{if } s>0,\ (x^*,y^*)\in\Gamma_k\end{cases}$$

where k^* is the random integer so that $(x_1^*,y_1^*)\in\Gamma_{k^*}$.

We then directly argue by using (4.5), (4.5) and (4.7) that

$$\begin{aligned} J(t,x,y,\hat c,\hat\alpha) &= \mathbb{E}\left[U(c^*)+\rho\, J\left(t-1,x_1^*,y_1^*,c^\epsilon(sx_{k^*},y_{k^*}),\alpha^\epsilon(x_{k^*},y_{k^*})\right)\right] \\ &\geq \mathbb{E}\left[U(c^*)+\rho\, J\left(t-1,x_{k^*},y_{k^*},c^\epsilon(x_{k^*},y_{k^*}),\alpha^\epsilon(x_{k^*},y_{k^*})\right)\right]-\epsilon \\ &\geq \mathbb{E}\left[U(c^*)+\rho\, v\,(t-1,x_{k^*},y_{k^*})\right]-2\epsilon \\ &\geq \mathbb{E}\left[U(c^*)+\rho\, v\left(t-1,x_1^*,y_1^*\right)\right]-3\epsilon. \end{aligned}$$

We now use the choice of $(c^*,\alpha^*)\in\mathbb{U}(x,y)$ and the inequality $v\geq J$ to arrive at

$$\begin{aligned} v(t,x,y) &\geq J(t,x,y,\hat c,\hat\alpha)\geq\mathbb{E}\left[U(c^*)+\rho\, v\left(t-1,x_1^*,y_1^*\right)\right]-3\epsilon \\ &\geq \sup_{(c,\alpha)\in\mathbb{U}(x,y)}\mathbb{E}\left[U(c)+\rho\, v\,(t-1,x_1,y_1)\right]-4\epsilon.\quad\Box \end{aligned}$$

5. Optimal strategies

In this section, we show that the value function is the unique solution of (4.1) and the maximizers of the expression in (4.1) form an optimal feedback policy. To formalize the final statement, we assume the following about the structure of the set $\mathbb{U}(x,y)$ defined by (2.5). This assumption is verified under natural structural conditions in Lemma 3.

$$\text{For each } (x,y)\in\bar{\mathbb{L}},\qquad \mathbb{U}(x,y) \text{ is a bounded subset of } \mathbb{R}^+\times\mathbb{R}^N. \tag{5.1}$$

An immediate consequence of this assumption is that the value function is finite and the maximum is achieved as the problem becomes a finite step maximization over compact set. Hence (5.1) implies (3.3).

Since U and v are concave and $\mathbb{U}$ is compact under the above assumption, we see that for each $(x, y) \in \bar{\mathbb{L}}$ there are maximizers of the expression in (4.1). By concavity, and hence continuity, of U and v, we conclude that there exists

$$\nu^* := (C^*, A^*) : \{1, \ldots, T\} \times \bar{\mathbb{L}} \to \mathbb{R}^+ \times \mathbb{R}^N,$$

so that

$$\nu^*(t, x, y) \in \mathbb{U}(x, y), \quad \forall\, (x, y) \in \bar{\mathbb{L}}, \tag{5.2}$$

and again for every $(x, y) \in \bar{\mathbb{L}}$ and $k \in \{1. \ldots, T\}$,

$\nu^*(t, x, y)$ maximizes the expression in (4.1) at time t.

We may now solve (2.2)–(2.3) starting from any initial point $(x, y) \in \bar{\mathbb{L}}$ and using the feedback strategy ν^*. Indeed, for a given initial data $(x, y) \in \bar{\mathbb{L}}$ and a horizon $t \in \{1, \ldots, T\}$, let (x^t, y^t) be the solution,

$$x^t_{k+1} = \left(x^t_k - A^*(t-k, x^t_k, y^t_k) \cdot \vec{1} - g(A^*(t-k, x^t_k, y^t_k)) - C^*(t-k, x^t_k, y^t_k)\right)(1+r),$$

$$y^*_{k+1} = (y^*_k + A^*(t-k, x^t_k, y^t_k))(1 + R_{k+1}),$$

for $k \in \{0, 1, \ldots, t-1\}$ together with initial condition $(x^t_0, y^t_0) = (x, y)$. Then, the actual open loop controls are obtained by simply evaluating ν^* at $(t-k, x^t_k, y^t_k)$, i.e.,

$$c^t_k = C^*(t-k, x^t_k, y^t_k), \qquad \alpha^t_k = A^*(t-k, x^t_k, y^t_k), \quad k = 0, 1, \ldots, t-1.$$

This procedure constructs the controls (c^t_k, α^t_k) for times $k = 0, 1, \ldots, t-1$. Moreover, by (5.2),

$$(c^t_k, \alpha^t_k) \in \mathbb{U}(x^t_k, y^t_k), \quad k = 0, 1, \ldots, t-1.$$

We then extend (c^t_k, α^t_k) for all k's as in Remark 2. The resulting extension, called (c^t, α^t), is admissible, i.e., $(c^t, \alpha^t) \in \mathbb{A}(x, y)$.

Theorem 1. *Assume (5.1). Then the value function is the unique continuous, solution of Eq. (4.1). Moreover, the feedback control (c^*, α^*) constructed above is optimal.*

Proof. We have shown that the value function is a solution of the dynamic programming equation (4.1). Let (c^t, α^t) be the admissible control processes constructed above. Since $\nu^*(t, x, y)$ maximizes the expression in (4.1) at time t,

$$v(t, x, y) = \mathbb{E}\left[U(c^t_0) + \rho v(t-1, x^t_1, y^t_1)\right].$$

Using the construction in the proof of Proposition 1, and we iterate this to arrive at

$$v(t, x, y) = \mathbb{E}\left[\sum_{k=0}^{t-1} \rho^k U(c^t_k) + \rho^t v(0, x^t_t, y^t_t)\right].$$

We now use the initial condition $v(0, x, y) = \widehat{U}(L(x, y))$ to conclude that

$$v(t, x, y) = \mathbb{J}(t, x, y, c^*, \alpha^*).$$

To prove uniqueness, let V be a continuous solution of the dynamic programming equation (4.1) with the initial condition $V(0, x, y) = \widehat{U}(L(x, y))$. We construct a feedback control $\bar{\eta}$ exactly as η^t and an open loop control $(\bar{c}, \bar{\alpha})$ again exactly as (c^t, α^t). Then, the preceding calculations also show that

$$V(t, x, y) = \mathbb{J}(t, x, y, \bar{c}, \bar{\alpha}) \le v(t, x, y).$$

To prove the opposite inequality, let $\eta = (c, \alpha)$ be any admissible control in $\mathbb{A}(x, y)$. Then, by (2.6),

$$(c_k, \alpha_k) \in \mathbb{U}(x_k, y_k), \quad k = 0, 1, \ldots, t-1,$$

where (x_k, y_k) is the solution of (2.2)–(2.3) with initial condition (x, y) and control η. We use the dynamic programming equation (and the construction given in Proposition 1) to conclude that

$$V(k, x_k, y_k) \ge \mathbb{E}\left[U(c_k) + \rho V(k-1, x_{k+1}, y_{k+1}) \mid \mathrm{F}_k\right], \quad k = 0, 1, \ldots, t-1.$$

We now iterate this inequality. The result is

$$\begin{aligned} V(t, x, y) &\ge \mathbb{E}\left[U(c_0) + \rho V(t-1, x_1, y_1)\right] \\ &\ge \mathbb{E}\left[U(c_0) + \rho \mathbb{E}\left[U(c_1) + \rho V(t-2, x_2, y_2) \mid \mathrm{F}_1\right]\right] \\ &= \mathbb{E}\left[U(c_0) + \rho U(c_1) + \rho^2 V(t-2, x_2, y_2)\right] \\ &\ge \cdots \\ &\ge \mathbb{E}\left[\sum_{k=0}^{t-1} \rho^k U(c_k) + \rho^k V(0, x_t, y_t)\right] \\ &= \mathbb{J}(t, x, y, c, \alpha), \end{aligned}$$

where in the final step we use the initial condition. Since $(c, \alpha) \in \mathbb{A}(x, y)$ is arbitrary, the above implies that

$$V(t, x, y) \geq \sup_{(c,\alpha)\in\mathbb{A}(x,y)} \mathbb{J}(t, x, y, c, \alpha) = v(t, x, y). \quad \square$$

We continue by providing a natural sufficient condition for (5.1). For $I \subset \{1, \ldots, N\}$ and $\delta > 0$, define

$$\Omega^{\delta,I} := \{R_1^i \leq r - \delta, \text{ for } i \in I, \text{ and } R_1^j \geq r + \delta, \text{ for } j \notin I\}.$$

Recall that μ is the probability measure associated with returns R_k.

Lemma 3. *Assume (3.3) and that for some $\delta > 0$,*

$$\mu\left(\Omega^{\delta,I}\right) > 0,$$

for every subset $I \subset \{1, \ldots, N\}$. Then, $\mathbb{U}(x, y)$ is a bounded subset of $\mathbb{R}^+ \times \mathbb{R}^N$ for all $(x, y) \in \bar{\mathbb{L}}$.

Proof. It is clear that if $(c, \alpha) \in \mathbb{U}(x, y)$, then c must be bounded by above. Now suppose that there are $(c^m, \alpha^m) \in \mathbb{U}(x, y)$ so that $|\alpha^m|$ tends to infinity. Considering a subsequence, we may assume that all components of α^m converge (including the limit points $\pm\infty$). First assume that $(\alpha^m)^i$ converges to plus infinity for some i. Set I to be the set of indices for which the limit point is plus infinity. Then, one can argue that on the set $\Omega^{\delta,I}$,

$$L\left((x - \alpha \cdot \vec{1} - g(\alpha) - c)(1 + r), (y + \alpha)(1 + R_1)\right)$$

converges to minus infinity. Hence a contradiction to the fact that $(c^m, \alpha^m) \in \mathbb{U}(x, y)$ and thus the above expression is non-negative with probability one.

Now, if $(\alpha^m)^i$ converges to minus infinity for some i, we set I to be the complement of the set on which the limit point is minus infinity and argue similarly. $\square$

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