4. For a two-period binomial model, you are given:

(i) Each period is one year.

(ii) The current price for a nondividend-paying stock is 20.

(iii) $u = 1.2840$, where u is one plus the rate of capital gain on the stock per period if the stock price goes up.

(iv) $d = 0.8607$, where d is one plus the rate of capital loss on the stock per period if the stock price goes down.

(v) The continuously compounded risk-free interest rate is 5%.

Calculate the price of an American call option on the stock with a strike price of 22.

(A) 0

(B) 1

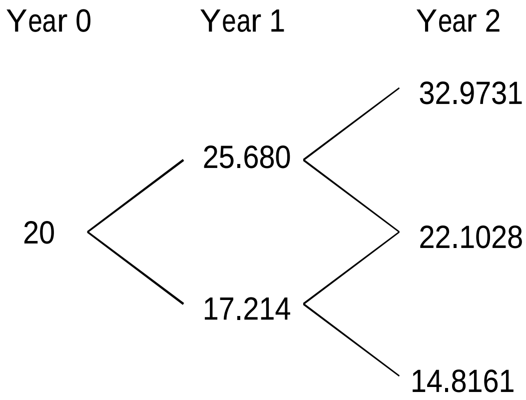
(C) 2

(D) 3

(E) 4

Solution to (4) Answer: (C)

First, we construct the two-period binomial tree for the stock price.



The calculations for the stock prices at various nodes are as follows:

$S_u = 20 \times 1.2840 = 25.680$
$S_d = 20 \times 0.8607 = 17.214$
$S_{uu} = 25.68 \times 1.2840 = 32.9731$
$S_{ud} = S_{du} = 17.214 \times 1.2840 = 22.1028$
$S_{dd} = 17.214 \times 0.8607 = 14.8161$

The risk-neutral probability for the stock price to go up is

$$p^* = \frac{e^{rh} - d}{u - d} = \frac{e^{0.05} - 0.8607}{1.2840 - 0.8607} = 0.4502.$$

Thus, the risk-neutral probability for the stock price to go down is 0.5498.

If the option is exercised at time 2, the value of the call would be
$C_{uu} = (32.9731 - 22)_+ = 10.9731$
$C_{ud} = (22.1028 - 22)_+ = 0.1028$
$C_{dd} = (14.8161 - 22)_+ = 0$

If the option is European, then $C_u = e^{-0.05}[0.4502C_{uu} + 0.5498C_{ud}] = 4.7530$ and
$C_d = e^{-0.05}[0.4502C_{ud} + 0.5498C_{dd}] = 0.0440$.
But since the option is American, we should compare C_u and C_d with the value of the option if it is exercised at time 1, which is 3.68 and 0, respectively. Since 3.68 < 4.7530 and 0 < 0.0440, it is not optimal to exercise the option at time 1 whether the stock is in the up or down state. Thus the value of the option at time 1 is either 4.7530 or 0.0440.

Finally, the value of the call is
$C = e^{-0.05}[0.4502(4.7530) + 0.5498(0.0440)] = 2.0585.$

Remark: Since the stock pays no dividends, the price of an American call is the same as that of a European call. See pages 294-295 of McDonald (2006). The European option price can be calculated using the binomial probability formula. See formula (11.17) on page 358 and formula (19.1) on page 618 of McDonald (2006). The option price is

$$e^{-r(2h)}\left[\binom{2}{2}p^{*2}C_{uu} + \binom{2}{1}p^*(1-p^*)C_{ud} + \binom{2}{0}(1-p^*)^2C_{dd}\right]$$
$$= e^{-0.1}\,[(0.4502)^2\times 10.9731 + 2\times 0.4502\times 0.5498\times 0.1028 + 0]$$
$$= 2.0507$$

5. Consider a 9-month dollar-denominated American put option on British pounds. You are given that:

(i) The current exchange rate is 1.43 US dollars per pound.

(ii) The strike price of the put is 1.56 US dollars per pound.

(iii) The volatility of the exchange rate is $\sigma = 0.3$.

(iv) The US dollar continuously compounded risk-free interest rate is 8%.

(v) The British pound continuously compounded risk-free interest rate is 9%.

Using a three-period binomial model, calculate the price of the put.

Solution to (5)

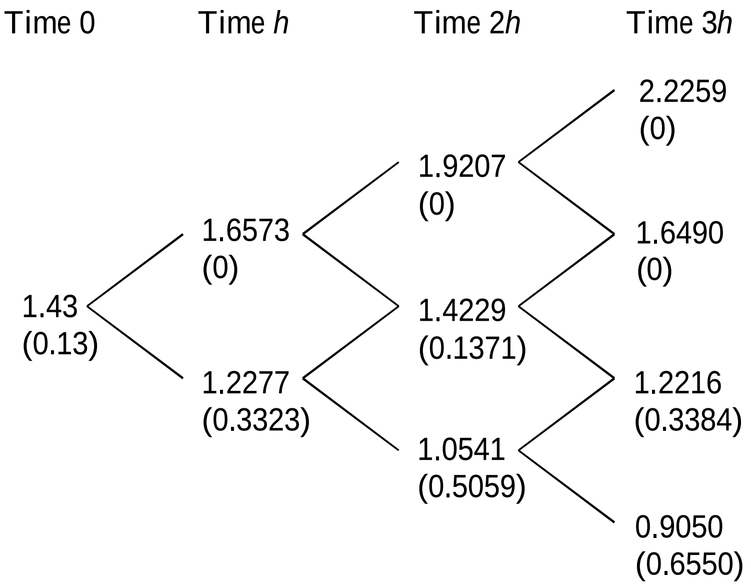
Each period is of length h = 0.25. Using the first two formulas on page 332 of McDonald (2006):

$$u = \exp[-0.01 \times 0.25 + 0.3 \times \sqrt{0.25}\,] = \exp(0.1475) = 1.158933,$$
$$d = \exp[-0.01 \times 0.25 - 0.3 \times \sqrt{0.25}\,] = \exp(-0.1525) = 0.858559.$$

Using formula (10.13), the risk-neutral probability of an up move is

$$p^* = \frac{e^{-0.01 \times 0.25} - 0.858559}{1.158933 - 0.858559} = 0.4626.$$

The risk-neutral probability of a down move is thus 0.5374. The 3-period binomial tree for the exchange rate is shown below. The numbers within parentheses are the payoffs of the put option if exercised.



The payoffs of the put at maturity (at time 3h) are
$P_{uuu} = 0$, $P_{uud} = 0$, $P_{udd} = 0.3384$ and $P_{ddd} = 0.6550$.

Now we calculate values of the put at time 2h for various states of the exchange rate.

If the put is European, then
$P_{uu} = 0$,
$P_{ud} = e^{-0.02}[0.4626P_{uud} + 0.5374P_{udd}] = 0.1783$,
$P_{dd} = e^{-0.02}[0.4626P_{udd} + 0.5374P_{ddd}] = 0.4985$.
But since the option is American, we should compare P_{uu}, P_{ud} and P_{dd} with the values of the option if it is exercised at time 2h, which are 0, 0.1371 and 0.5059, respectively. Since 0.4985 < 0.5059, it is optimal to exercise the option at time 2h if the exchange rate has gone down two times before. Thus the values of the option at time 2h are $P_{uu} = 0$, $P_{ud} = 0.1783$ and $P_{dd} = 0.5059$.

Now we calculate values of the put at time h for various states of the exchange rate.

If the put is European, then
$P_u = e^{-0.02}[0.4626P_{uu} + 0.5374P_{ud}] = 0.0939$,
$P_d = e^{-0.02}[0.4626P_{ud} + 0.5374P_{dd}] = 0.3474$.
But since the option is American, we should compare P_u and P_d with the values of the option if it is exercised at time h, which are 0 and 0.3323, respectively. Since 0.3474 > 0.3323, it is not optimal to exercise the option at time h. Thus the values of the option at time h are P_u = 0.0939 and P_d = 0.3474.

Finally, discount and average P_u and P_d to get the time-0 price,

$$P = e^{-0.02}[0.4626P_u + 0.5374P_d] = 0.2256.$$

Since it is greater than 0.13, it is not optimal to exercise the option at time 0 and hence the price of the put is 0.2256.

Remarks:

(i) Because $\dfrac{e^{(r-\delta)h} - e^{(r-\delta)h-\sigma\sqrt{h}}}{e^{(r-\delta)h+\sigma\sqrt{h}} - e^{(r-\delta)h-\sigma\sqrt{h}}} = \dfrac{1-e^{-\sigma\sqrt{h}}}{e^{\sigma\sqrt{h}} - e^{-\sigma\sqrt{h}}} = \dfrac{1}{1+e^{\sigma\sqrt{h}}}$, we can also calculate the risk-neutral probability p^* as follows:

$$p^* = \frac{1}{1+e^{\sigma\sqrt{h}}} = \frac{1}{1+e^{0.3\sqrt{0.25}}} = \frac{1}{1+e^{0.15}} = 0.46257.$$

(ii) $$1 - p^* = 1 - \frac{1}{1+e^{\sigma\sqrt{h}}} = \frac{e^{\sigma\sqrt{h}}}{1+e^{\sigma\sqrt{h}}} = \frac{1}{1+e^{-\sigma\sqrt{h}}}.$$

(iii) Because $\sigma > 0$, we have the inequalities

$$p^* < ½ < 1 - p^*.$$