King Saud University	College of Sciences	Mathematics Department
MATH254	Exam I, S2/37-38H	Duration: 90 Minutes

Q.1[5 marks]

Use the bisection method to find the third approximation of $\sqrt[3]{2}$ starting with the initial interval $[1,2]$, and find the corresponding absolute error. Also, compute the number of iterations needed to achieve an approximation accurate to within 10^{-5}.

Q.2[5 marks]

Consider the equation: $3x^2 - e^x = 0$, which has a root in $[0.5, 1.5]$. Determine which of the following iterations is suitable for computing this root; (justify your answer)

$$(i)\quad x_{n+1} = \sqrt{\frac{e^{x_n}}{3}} \qquad (ii)\quad x_{n+1} = \ln(3) + 2\ln(x_n), \quad n = 0, 1, 2, \ldots.$$

Then, use the suitable one to compute the second approximation of the root using $x_0 = 1$, and find an upper bound for the corresponding error.

Q.3[5 marks]

The equation: $1 - 2\cos(x) + \cos^2(x) = 0$, has the root $\alpha = 0$. Develop Newton's formula for computing this root, then use it to find the second approximation starting with $x_0 = 0.5$. Also, find the order of convergence of your formula.

Q.4[5 marks]

The equation: $1 - x + \ln(x) = 0$, has the root $\alpha = 1$. Staring with $x_0 = 0.5$, compute the second approximation of this root using a quadratic convergent method, and find the corresponding absolute error.

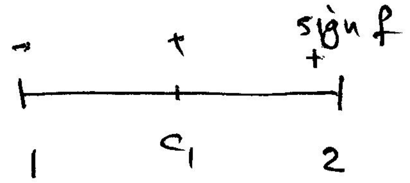
Q.5[5 marks]

Consider the nonlinear system;

$$\begin{aligned} x^3 + 3y^2 &= 21, \\ x^2 + 2y + a &= 0. \end{aligned}$$

Suppose that applying Newton's method to this system starting with the initial approximation $(x_0, y_0)^T = (1, -1)^T$ gives $(x_1, y_1)^T = (2.5556, -3.0556)^T$.
Find the value of a.

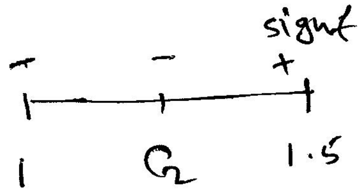
Q.1 Let $x = \sqrt[3]{2} \iff x^3 = 2 \iff f(x) := x^3 - 2$.



$a_1 = 1 \quad , \quad b_1 = 2$

$$\Rightarrow c_1 = \frac{a_1 + b_1}{2} = \frac{1+2}{2} = 1.5 \qquad 1$$

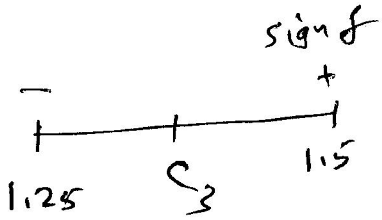
$$f(c_1) = f(1.5) = 1.375 > 0$$



$\therefore \; a_2 = 1 \quad , \quad b_2 = 1.5$

$$\Rightarrow c_2 = \frac{a_2 + b_2}{2} = \frac{1+1.5}{2} = 1.25 \qquad 1$$

$$f(c_2) \approx -0.04688 < 0$$



$$\therefore \; a_3 = 1.25 \quad , \quad b_3 = 1.5 \qquad 1$$

$$\Rightarrow c_3 = \frac{a_3 + b_3}{2} = \frac{1.25 + 1.5}{2} = 1.375.$$

$$\text{Absolute error} = \left| \sqrt[3]{2} - c_3 \right| \simeq \left| 1.259921 - 1.375 \right|$$

$$\approx 0.115079 \qquad 1$$

$n = ?$ so that $|\alpha - c_n| \le 10^{-5}$.

But $|\alpha - c_n| \le \dfrac{b-a}{2^n}$

$\therefore$ put $\dfrac{b-a}{2^n} \le 10^{-5}$ and find n

$$\Rightarrow \frac{2-1}{2^n} \le 10^{-5} \iff 2^n \ge 10^5$$

$$\Rightarrow n \ge \frac{5 \ln 10}{\ln 2} \approx 16.61$$

$$\therefore \; n = 17. \qquad 1$$

Q.2 (i) $g_1(x) = \sqrt{\frac{e^x}{3}} = \frac{e^{\frac{1}{2}x}}{\sqrt{3}}$, $x \in [0.5, 1.5]$

(1) g_1 is cont. on $[0.5, 1.5]$, since $\frac{e^x}{3} > 0 \ \forall x \in [0.5, 1.5]$

(2) $g_1'(x) = \frac{e^{\frac{1}{2}x}}{2\sqrt{3}} > 0$, $\forall x \in [0.5, 1.5]$

$\Rightarrow g_1$ is increasing on $[0.5, 1.5]$

$\Rightarrow \quad g(0.5) \leq g(x) \leq g(1.5)$

$0.5 < 0.741332 \leq g(x) \leq 1.22225 < 1.5$

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$\Rightarrow g(x) \in [0.5, 1.5) \ \forall x \in [0.5, 1.5]$.

(3) g_1' is increasing on $[0.5, 1.5)$

$\Rightarrow \max_{[0.5, 1.5)} |g_1'(x)| = |g_1'(1.5)| \approx \underbrace{0.611125}_{k} < 1$

$\therefore$ The first iteration is suitable.

(ii) $g_2(x) = \ln 3 + 2 \ln x$, $x \in [0.5, 1.5]$

(1) g_2 is cont. on $[0.5, 1.5]$

1

(2) But $g_2(0.5) = \ln 3 + 2 \ln 0.5$
$\cong -0.2877 \notin [0.5, 1.5]$

$\therefore$ The second iteration is not suitable.

Using the first iteration with $x_0 = 1$ we get.

$$x_0 = 1 \Longrightarrow x_1 = g(x_0) = g(1) = \sqrt{\frac{e}{3}} \approx 0.95189 \quad (1)$$

$$x_2 = g_1(x_1) = g_1(0.95189) \cong 0.929265$$

Error bound formula is:

$$|\alpha - x_n| \le \frac{k^n}{1-k} |x_1 - x_0| ,$$

$$\Rightarrow |\alpha - x_2| \le \frac{(0.611125)^2}{1 - 0.611125} |0.95189 - 1| \quad (1)$$

$$\le 0.0462046$$

Q.3 $f(x) = 1 - 2\cos x + \cos^2 x$

$$= (1 - \cos x)^2$$

$$\Rightarrow f'(x) = 2 \sin x (1 - \cos x)$$

$\therefore$ Newton's formula is:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad (1)$$

$$= x_n - \frac{(1-\cos x_n)^2}{2\sin x_n (1 - \cos x_n)} = x_n - \frac{1 - \cos x_n}{2 \sin x_n},$$

$n = 0, 1, \ldots$

$$x_0 = 0.5 \Longrightarrow x_1 = x_0 - \frac{1-\cos x_0}{2\sin x_0} \approx 0.37233, \quad 1$$

$$x_2 = x_1 - \frac{1-\cos x_1}{2\sin x_1} \approx 0.27816. \quad 1$$

Newton's formula is on the form $x_{n+1} = g(x_n)$, where

$$g(x) = x - \frac{1-\cos x}{2\sin x}$$

$$\Rightarrow g'(x) = 1 - \frac{1}{2}\,\frac{\sin^2 x - (1-\cos x)\cos x}{\sin^2 x}$$

$$= 1 - \frac{1}{2}\left(\frac{\sin^2 x - \cos x + \cos^2 x}{\sin^2 x}\right)$$

$$= 1 - \frac{1}{2}\left(\frac{1-\cos x}{\sin^2 x}\right)$$

$$g'(0) = 1 - \frac{1}{2}\lim_{x\to 0}\frac{1-\cos x}{\sin^2 x} \quad 1 \quad \frac{0}{0}$$

$$= 1 - \frac{1}{2}\lim_{x\to 0}\frac{\cancel{\sin x}}{2\cancel{\sin x}\cos x}$$

$$= 1 - \frac{1}{2} = \frac{1}{2} \neq 0 \quad 1$$

$\Longrightarrow$ the method converges linearly.

Q.4 $f(x) = 1 - x + \ln x$, $\alpha = 1$

$f'(x) = -1 + \frac{1}{x} \Rightarrow f(\alpha) = f(1) = 0$ 1

$f''(x) = \frac{-1}{x^2} \Rightarrow f''(\alpha) = f''(1) \neq 0$

$\Rightarrow$ The root $\alpha = 1$ has multiplicity $m = 2$. 1

$\therefore$ The quadratically convergent method is the modified Newton's method which is:

$$x_{n+1} = x_n - \frac{m f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{2[1 - x_n + \ln x_n]}{\left[-1 + \frac{1}{x_n}\right]} \quad , n = 0, 1, \cdots$$

$x_0 = 0.5 \Rightarrow x_1 \cong 0.8862944$ 1

$\therefore x_2 \cong 0.995427$ 1

Absolute error is

$$|\alpha - x_2| = |1 - 0.995427| \simeq 0.0045727$$ 1

Q.5. $f(x,y) = x^3 + 3y^2 - 21 \Rightarrow f_x = 3x^2, \; f_y = 6y$ 1

$g(x,y) = x^2 + 2y + a \Rightarrow g_x = 2x, \; g_y = 2$

At $(1,-1)$

$f = -17, \quad f_x = 3, \quad f_y = -6$ 1

$g = a-1, \quad g_x = 2, \quad g_y = 2$

Newton's formula is

$$\underline{X}^{(n+1)} = \underline{X}^{(n)} - J^{-1}[x_n]\, \underline{F}(X_n), \quad \underline{X}^{(0)} = [1,-1]^T$$ 1

$$\begin{pmatrix} 2.5556 \\ -3.0556 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 3 & -6 \\ 2 & 2 \end{pmatrix}^{-1} \begin{pmatrix} -17 \\ a-1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 3 & -6 \\ 2 & 2 \end{pmatrix}^{-1} \begin{pmatrix} -17 \\ a-1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 2.5556 \\ -3.0556 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -17 \\ a-1 \end{pmatrix} = \begin{pmatrix} 3 & -6 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} -1.5556 \\ 2.0556 \end{pmatrix} \simeq \begin{pmatrix} -17 \\ 1 \end{pmatrix}$$ 1

$$\Rightarrow a-1 = 1 \Rightarrow a = 2$$ 1