Midterm

King Saud University,
College of Sciences
Mathematical Department.

Mid-Term 2/S1/2011
Full Mark:40. Time 1H30mn
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**Question 1**[6,6]. Given the initial value problem

$$\begin{cases} (x-1)y'' - xy' + y = 1 \\ y(2) = 0, \; y'(2) = 1. \end{cases} \qquad (*)$$

**a)** Find the largest interval for which the initial value problem (*) has a unique solution.

**b)** If $y_1 = e^x$ is a solution for the homogeneous equation $(x-1)y'' - xy' + y = 0$, then by using the method of reduction of order find its second solution $y_2$.

**Question 2**[6,6]. **a)** The motion of a spring-mass system is discribed by

$$\frac{d^2y}{dt^2} + 16y = 0,$$

where $y$ is the displacement and $t$ is time. Determine the solution satisfying $y(0) = 1$, and $y'(0) = 8$.

**b)** By using the undetermined coefficients method, give only the form of the particular solution $y_p$ of the differential equation

$$y^{(3)} - 3y'' + 3y' - y = 8 - 2x + x^2e^x.$$

**Question 3**[7]. Solve the nonhomogeneous differential equation

$$x^2y'' - xy' + y = 2x, \qquad x > 0.$$

**Question 4**[9]. Solve the following linear system of differential equations

$$\begin{cases} 4y'' - 4x = 1 \\ x'' - y = t^3. \end{cases}$$

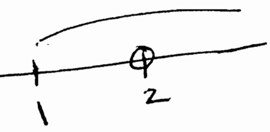
## Answer Sheet

## Exam 2 / S1 / 2011

**Answer to Q1**

a) $a_2(x)=x-1$, $a_1(x)=-x$, $a_0(x)=1$, $f(x)=1$ are continuous on $\mathbb{R}=(-\infty,\infty)$. 3

$a_2(x)=x-1=0 \Rightarrow x=1$.



Since $x_0=2$, then the largest interval for which the given I.V.P. (*) has a unique solution is $(1,\infty)$ ~~3~~

b) $y=e^{x}u(x) \Rightarrow y'=e^{x}(u'+u)$, $y''=e^{x}(u''+2u'+u)$.

Thus $(x-1)y''-xy'+y=(x-1)e^{x}(u''+2u'+u)-xe^{x}(u'+u)+e^{x}u=0$ 2

$\Leftrightarrow (x-1)u''+(2x-2-x)u'=0$

Let $v=u'$, then

$(x-1)v'+(x-2)v=0 \Rightarrow \frac{dv}{v}=\frac{2-x}{x-1}dx$

$\Rightarrow \ln\left|\frac{v}{c}\right|=-(x-\ln|x-1|)$ 2

$\Rightarrow v(x)=c_1e^{-x}(x-1)$

Hence $u'=c_1e^{-x}(x-1) \Rightarrow u(x)=c_1\int(x-1)e^{-x}dx+c_2$

$\Rightarrow u(x)=-xe^{-x} \Rightarrow \boxed{y_2=-x}$ with $c_1=1$, $c_2=0$ 2

Answer to Q2

a) $y'' + 16y = 0$

Charact Eq: $m^2 + 16 \Rightarrow m_1 = 4i;\ m_2 = -4i$ (2)

$y_{gh} = C_1 \cos 4t + C_2 \sin 4t$ (2)

$y(0) = 1 \Rightarrow 1 = C_1$

$y' = -4C_1 \sin 4t + 4C_2 \cos 4t$

$y'(0) = 8 \Rightarrow 4C_2 = 8 \Rightarrow C_2 = 2$ (2)

Thus $y_p = \cos 4t + 2 \sin 4t$

b) Charact Eq: $m^3 - 3m^2 + 3m - 1 = 0,\ m_1 = 1$ (2)

$m^3 - 3m^2 + 3m - 1 = (m-1)(m^2 - 2m + 1) = 0$

$m^2 - 2m + 1 = 0 \Leftrightarrow (m-1)^2 = 0 \Rightarrow m_2 = m_3 = 1$

$m = 1$ is a root of order of multiplicity 3 (triple root)

Thus $y_p = (Ax + B) + x^{s}(Cx^2 + Dx + E)e^{x}$ (4)

$s = 3$ since $r = 1$ is a triple root

Hence $y_p = (Ax + B) + (Cx^5 + Dx^4 + Ex^3)e^{x}$

Answer to Q3 $\quad x^2 y'' - x y' + y = 2x, \quad x > 0 \quad$ ( Cauchy Euler Eq )

$$y = x^m$$

Charat Eq $\quad m^2 - 2m + 1 = 0 \Rightarrow (m-1)^2 = 0$

$$y_h = C_1 x + C_2 x \ln x \qquad 2$$

Using the method of variation of parameters, we have

with $y_p = C_1(x) x + C_2(x) x \ln x$

$$\begin{cases} C_1' x + C_2' x \ln x = 0 \\ C_1' + C_2' (\ln x + 1) = \frac{2}{x} \end{cases}$$

$$\Delta = W = \begin{vmatrix} x & x \ln x \\ 1 & \ln x + 1 \end{vmatrix} = x$$

$$C_1'(x) = \frac{\begin{vmatrix} 0 & x \ln x \\ \frac{2}{x} & \ln x + 1 \end{vmatrix}}{x} = -\frac{2 \ln x}{x} \qquad 2$$

$$\Rightarrow C_1(x) = -2 \int \frac{\ln x}{x} dx$$

Let $u = \ln x \Rightarrow du = \frac{dx}{x}$

$$C_1(x) = -2 \int u \, du = \frac{-2u^2}{2} = -u^2 = -(\ln x)^2$$

$$C_2'(x) = \frac{\begin{vmatrix} x & 0 \\ 1 & \frac{2}{x} \end{vmatrix}}{x} = \frac{2}{x} \Rightarrow C_2(x) = 2 \ln x \qquad 2$$

Hence $y_p = -(\ln x)^2 x + 2x(\ln x)^2 = x(\ln x)^2$

$$y_g = y_h + y_p = C_1 x + C_2 x \ln x + x(\ln x)^2 \qquad 1$$

Answer to Q4 : $\begin{cases} 4y'' - 4x = 1 \\ x'' - y = t^3 \end{cases}$

operator form $\begin{cases} 4D^2[y] - 4x = 1 \rightarrow (1) \\ D^2[x] - y = t^3 \rightarrow (2) \end{cases}$ 2

We apply $4D^2$ to (2), we get

$4D^4[x] - 4D^2[y] = 24t \rightarrow (3)$

(1) + (3) yields

$4x^{(4)} - 4x = 1 + 24t \rightarrow (3)$ 2

Ch Eq: $4m^4 - 4 = 0 \Rightarrow m^4 - 1 = 0 = (m^2-1)(m^2+1) \Rightarrow$

$m_1 = 1, \; m_2 = -1, \; m_3 = i, \; m_4 = -i$

Hence $x_{gh}(t) = C_1 e^t + C_2 e^{-t} + C_3 \cos t + C_4 \sin t$

$x_p = At + B, \; x_p' = A, \; x_p'' = 0$

Hence $-4At - 4B = 1 + 24t$

$\Rightarrow A = -6, \; B = -\frac{1}{4}$ 2

$\Rightarrow x_p(t) = -6t - \frac{1}{4}$

Hence $x(t) = C_1 e^t + C_2 e^{-t} + C_3 \cos t + C_4 \sin t - 6t - \frac{1}{4}$

$x'(t) = C_1 e^t - C_2 e^{-t} - C_3 \sin t + C_4 \cos t - 6$

$x''(t) = C_1 e^t + C_2 e^{-t} - C_3 \cos t - C_4 \sin t$ 2

Thus $y(t) = C_1 e^t + C_2 e^{-t} - C_3 \cos t - C_4 \sin t - t^3$

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