King Saud University
Mathematics Department

Math-254 **(Second Semester, 1437-1438)** **Second Mid Term Exam**

FULL MARKS: 25 TIME 90 mins

Q 1[5]
Use simple Gauss elimination method to find values of α for which the following system is consistent

$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & - & 3x_3 & = & 4 \\ 3x_1 & - & x_2 & + & 5x_3 & = & 2 \\ 4x_1 & + & x_2 & + & (\alpha^2 - 14)x_3 & = & \alpha + 2. \end{array}$$

Also write the solution for $\alpha = 1$.

Q 2[5]
Discuss the conditioning of the linear system

$$\begin{array}{rcrcl} \frac{1}{2}x_1 & + & \frac{1}{3}x_2 & = & \frac{1}{63} \\ \frac{1}{3}x_1 & + & \frac{1}{4}x_2 & = & \frac{1}{168}. \end{array}$$

If $X^* = [0.142, -0.166]^T$ is an approximate solution of the system, then estimate the relative error.

Q 3[5]
For what values of b the following matrix is singular using Crout's method

$$\begin{pmatrix} 2 & -4 & b \\ 2 & 4 & 3 \\ 4 & -2 & 5 \end{pmatrix}.$$

Q 4[5]
Find the values of a such that the convergence of Jacobi method for the following system is guaranteed

$$\begin{array}{rcrcrcr} 2x_1 & + & ax_2 & + & ax_3 & = & 1 \\ ax_1 & + & 2x_2 & + & ax_3 & = & -1 \\ ax_1 & + & ax_2 & + & 2x_3 & = & 1. \end{array}$$

If $a = \frac{1}{4}$ and $X^0 = [-1, 2, 3]^T$ then compute the number of iterations needed to get accuracy within 10^{-4}.

Q 5[5]
Use the quadratic Lagrange interpolating polynomial by selecting the best points form $x =$ 1.7, 1.8, 1.9, 2.1, 2.3, 2.4, 3.1, 4.2, ~~and $x =$~~ 5.9 on the function $f(x) = \sqrt{2x+1}$ to estimate $\sqrt{5}$. Also compute the error bound and absolute error.

Solution Mid 2

Ex 1:

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & \alpha^2-4 & \alpha+2 \end{array}\right) \xrightarrow[\substack{E_2 - 3E_1 \\ E_3 - 4E_1}]{\substack{m_{21}=3 \\ m_{31}=4}} \left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -7 & \alpha^2-2 & \alpha-14 \end{array}\right)$$

$$\xrightarrow[E_3 - E_2]{m_{32}=1} \left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & 0 & \alpha^2-16 & \alpha-4 \end{array}\right)$$

The last equation gives $(\alpha^2-16)x_3 = \alpha - 4$

If $\alpha = 4 \Leftrightarrow 0 = 0$ (Infinity many solutions)

If $\alpha = -4 \Leftrightarrow 0 = -8$ No solutions

If $\alpha \neq 4$, $\alpha \neq -4$ one solution

Then system is consistent if $\alpha \neq -4$

For $\alpha = 1$:

$$-15x_3 = -3 \Rightarrow x_3 = \frac{1}{5}$$

$$-7x_2 + 14x_3 = -10 \Rightarrow x_2 = \frac{64}{35}$$

$$x_1 + 2x_2 - 3x_3 = 4 \Rightarrow x_1 = \frac{47}{35}$$

Ex2 :

$$A = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{4} \end{pmatrix} \longrightarrow \|A\|_\infty = 5/6$$

$$A^{-1} = \begin{pmatrix} \frac{72}{4} & -\frac{72}{3} \\ -\frac{72}{3} & \frac{72}{2} \end{pmatrix} \longrightarrow \|A^{-1}\|_\infty = 60$$

$$K(A) = \|A\|_\infty \|A^{-1}\|_\infty = 50 \longrightarrow \text{Ill conditioned}$$

$$\|b\|_\infty = \frac{1}{63}$$

$$r = b - Ax^* = \begin{pmatrix} \frac{1}{63} \\ \frac{1}{168} \end{pmatrix} - \begin{pmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 0.142 \\ -0.166 \end{pmatrix} = \begin{pmatrix} \frac{13}{63000} \\ \frac{1}{8400} \end{pmatrix}$$

$$\|r\|_\infty = \frac{13}{63000} = 2,063492 \; 10^{-4}$$

$$\frac{\|x - x^*\|_\infty}{\|x\|_\infty} \leq K(A) \frac{\|r\|_\infty}{\|b\|_\infty} = \frac{13}{20} = 0,65$$

Ex 3

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & -4 & b \\ 2 & 4 & 3 \\ 4 & -2 & 5 \end{pmatrix}$$

$$\Big\downarrow \frac{E_1}{2}$$

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 & \frac{b}{2} \\ 2 & 4 & 3 \\ 4 & -2 & 5 \end{pmatrix}$$

$$\Big\downarrow \begin{matrix} E_2 - 2E_1 \\ E_3 - 4E_1 \end{matrix}$$

$$\begin{pmatrix} 2 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 & \frac{b}{2} \\ 0 & 8 & 3-b \\ 0 & 6 & 5-2b \end{pmatrix}$$

$$\Big\downarrow \frac{E_2}{8}$$

$$\begin{pmatrix} 2 & 0 & 0 \\ 2 & 8 & 0 \\ 4 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 & \frac{b}{2} \\ 0 & 1 & \frac{3-b}{8} \\ 0 & 6 & 5-2b \end{pmatrix}$$

$$\Big\downarrow E_3 - 6E_2$$

$$\underbrace{\begin{pmatrix} 2 & 0 & 0 \\ 2 & 8 & 0 \\ 4 & 6 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 1 & -2 & \frac{b}{2} \\ 0 & 1 & \frac{3-b}{8} \\ 0 & 0 & \frac{11}{4} - \frac{5b}{4} \end{pmatrix}}_{U}$$

Then the matrix is singular if $11 - 5b = 0 \iff b = \frac{11}{5}$

Ex4 :

*/ $T_J = D^{-1}(L+U) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 0 & a & a \\ a & 0 & a \\ a & a & 0 \end{pmatrix}$

$$= \begin{pmatrix} 0 & \frac{a}{2} & \frac{a}{2} \\ \frac{a}{2} & 0 & \frac{a}{2} \\ \frac{a}{2} & \frac{a}{2} & 0 \end{pmatrix}$$

$\|T_J\|_\infty = |a|$

Jacobi converges if $|a| < 1 \iff -1 < a < 1$

*/ for $a = \frac{1}{4}$, $\|T_J\|_\infty = \frac{1}{4}$

$$x_1^{(1)} = \frac{1}{2}\left[1 - \frac{1}{4}x_2^{(0)} - \frac{1}{4}x_3^{(0)}\right] = \frac{1}{2}\left(1 - \frac{2}{4} - \frac{3}{4}\right) = -\frac{1}{8}$$

$$x_2^{(1)} = \frac{1}{2}\left[-1 - \frac{1}{4}x_1^{(0)} - \frac{1}{4}x_3^{(0)}\right] = \frac{1}{2}\left(-1 + \frac{1}{4} - \frac{3}{4}\right) = -\frac{3}{4}$$

$$x_3^{(1)} = \frac{1}{2}\left[1 - \frac{1}{4}x_1^{(0)} - \frac{1}{4}x_2^{(0)}\right] = \frac{1}{2}\left(1 + \frac{1}{4} - \frac{2}{4}\right) = \frac{3}{8}$$

$$x^{(1)} = \left(-\frac{1}{8}, -\frac{3}{4}, \frac{3}{8}\right)$$

$$\|x^{(1)} - x^{(0)}\|_\infty = \left\| \begin{pmatrix} -9/8 \\ -11/4 \\ -\frac{21}{8} \end{pmatrix} \right\|_\infty = \frac{11}{4}$$

$$\frac{\|T_J\|_\infty^k}{1 - \|T_J\|_\infty} \|x^{(1)} - x^{(0)}\|_\infty \leq 10^{-4}$$

$$\iff \left(\frac{1}{4}\right)^k \leq \frac{3/4}{11/4} 10^{-4} = \frac{3}{11} 10^{-4}$$

$$k \geq \frac{\ln(3/11\ 10^{-4})}{} = 7{,}58$$

Ex7 :

by taking $2n+1 = 5 \Leftrightarrow n = 2$

Therefore the best points for quadratic polynomial are

$x_0 = 1.8$, $x_1 = 1.9$ and $x_2 = 2.1$

$$f(x) \simeq P_2(x) = f(x_0)\, L_0(x) + f(x_1)\, L_1(x) + f(x_2)\, L_2(x)$$

at $x = 2$ we obtain

$$\sqrt{5} = f(2) \cong P_2(2) = f(1.8)\, L_0(2) + f(1.9)\, L_1(2) + f(2.1)\, L_2(2)$$

$$L_0(2) = \frac{(2-1.9)(2-2.1)}{(1.8-1.9)(1.8-2.1)} = -\frac{1}{3}$$

$$L_1(2) = \frac{(2-1.8)(2-2.1)}{(1.9-1.8)(1.9-2.1)} = 1$$

$$L_2(2) = \frac{(2-1.8)(2-1.9)}{(2.1-1.8)(2.1-1.9)} = \frac{1}{3}$$

then $\sqrt{5} = f(2) \simeq P_2(2) = \sqrt{4.6} \times (-\frac{1}{3}) + \sqrt{4.8} + \sqrt{5.2} \cdot (\frac{1}{3})$
$= 2.236086827$

Absolute Error $= |\sqrt{5} - P_2(2)| = 1.88495\ 10^{-5}$

$$E_2(x) = \frac{f^{(3)}(\eta)}{3!}(x-x_0)(x-x_1)(x-x_2)$$

$$|E_2(2)| \leq \frac{M}{3!} |(2-1.8)(2-1.9)(2-2.1)| = 2.203461\ 10^{-5}$$

with $M = \max |f^{(3)}(\eta)| = \max |3(2\eta+1)^{-5/2}| = 0.066103838$