Question 1 [3+4]

(1) Prove that $\lim_{x \to 0} \frac{|x|}{x}$ does not exist.

(2) Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

Show that f is differentiable everywhere, but f' is not continuous at the origin.

Question 2 [4+4]

(1) Show that $f : (0,1] \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$ is not uniformly continuous on $(0,1]$.

(2) Show that the function $f(x) = |x|$ is uniformly continuous on $\mathbb{R}$, but is not differentiable at $x = 0$.

Question 3 [5] Show that $2x^3 + 3x^2 + 6x + 10$ has exactly one real zero.

Question 4 [5] Show that for all $x > 0$, $\sqrt{1+x} < 1 + \frac{1}{2}x$.