132 Math Midterm Exam

Name:

ID:

QUESTION 1: Fill in the planks in the following and Explain your answer:

$\frac{2}{2}$ a- For any proposition p, The truth value of the proposition $p \leftrightarrow p$ is __T__.

p	$p \leftrightarrow p$
T	T
F	T

$\frac{2}{2}$ b- If $p \vee q$ is true, then the truth value of $\neg p \rightarrow q$ is __T__.

p	q	$p \vee q$	$\neg p$	$\neg p \rightarrow q$
T	T	T	F	T
T	F	T	F	T
F	T	T	T	T
F	F	F	T	F

or $p \vee q \equiv \neg p \rightarrow q$ so if $p \vee q$ T then $\neg p \rightarrow q$ is T

$\frac{2}{2}$ c- The negation of the statement $[\forall x \in \mathbb{R}: x^2 \geq 0]$ is $\exists x \in \mathbb{R}, x^2 < 0$

$\neg[\forall x \in \mathbb{R}, x^2 \geq 0]$

$\equiv \exists x \in \mathbb{R}, \neg(x^2 \geq 0)$

$\equiv \exists x \in \mathbb{R}, x^2 < 0$

$\frac{2}{2}$ d- The inverse of the contrapositive of the proposition $p \rightarrow q$ is $q \rightarrow p$.

the contrapositive :- $\neg q \rightarrow \neg p$

the inverse of it :- $\neg(\neg q) \rightarrow \neg(\neg p) \equiv q \rightarrow p$

$\frac{2}{2}$ e- To prove that for any integer n, 2 divides $n^2 + n$ using proof by cases, we need to discuss two cases which are __n even__ and __n odd__.

because, n even $\rightarrow n = 2k \rightarrow n^2 + n = 4k^2 + 2k = 2(2k^2 + k)$ divisible by 2

n odd $\rightarrow n = 2k+1 \rightarrow n^2 + n = 4k^2 + 4k + 1 + 2k + 1 = 2(2k^2 + 3k + 1)$ divisible by 2.

$\frac{2}{2}$ f- The truth value of the statement $\exists x \in \{1,2,3,4\}, 2^x < x$ is __false__.

$P(1): 2^1 = 2 < 1$ is false

$P(2): 2^2 = 4 < 2$ is false

$P(3): 2^3 = 8 < 3$ is false

$P(4): 2^4 = 16 < 4$ is false

QUESTION 2:

a- without using truth tables, prove that

$$\neg(p \to r) \to \neg q \equiv (p \wedge q) \to r.$$

$\neg(p \to r) \to \neg q \equiv \neg(\neg p \vee r) \to \neg q$

$\equiv (p \wedge \neg r) \to \neg q$

$\equiv \neg(p \wedge \neg r) \vee \neg q$

$\equiv \neg p \vee r \vee \neg q$

$\equiv \neg p \vee \neg q \vee r$

$\equiv \neg(p \wedge q) \vee r$

$\equiv (p \wedge q) \to r$

$\frac{3}{3}$

b- Show that the statement "For every positive integer n, $n^2 \geq 2n$" is false.

$\frac{2}{2}$

by counterexample,

there exists $n = 1$ where $1 \not\geq 2(1) = 2$, thus $P(1)$ is false

and $\forall n\, P(n)$ is false.

c- Prove that there exists an integer m such that $m^2 > 10^{100}$.

$\frac{1}{1}$

let $m = 11^{50}$, then $m^2 = (11^{50})^2 = 11^{100} > 10^{100}$ because

$11 > 10$.

QUESTION 3:

a- Prove that if n is an integer, then n is even if and only if $3n^2 + 2n + 1$ is odd.

$\rightarrow$) n even $\overset{?}{\rightarrow}$ $3n^2+2n+1$ is odd.

$\frac{5}{5}$

let n be even, then $\exists k \in \mathbb{Z}$ such that $n=2k$, thus;

$3n^2+2n+1 = 3(4k^2)+2(2k)+1 = 12k^2+4k+1 = 2(6k^2+2k)+1 = 2s+1$, $s=6k^2+2k\in\mathbb{Z}$

so $3n^2+2n+1$ is odd,

($\leftarrow$) by using contraposition, it is enough to prove that if n odd $\rightarrow$ $3n^2+2n+1$ is even,

let n be odd, then there exists $k\in\mathbb{Z}$ such that $n=2k+1$, so

$3n^2+2n+1 = 3(2k+1)^2+2(2k+1)+1 = 3(4k^2+4k+1)+4k+2+1 = 12k^2+12k+3+4k+3$

$= 12k^2+16k+6 = 2(6k^2+8k+3) = 2s$, $s = 6k^2+8k+3 \in \mathbb{Z}$

so $3n^2+2n+1$ is even,

Hence n even $\leftrightarrow$ $3n^2+2n+1$ is odd,

b- Prove that $\forall n \in A(3^n < n^2)$ is true, where $A = \{-1,-2,-3\}$.

$\frac{3}{3}$

$P(-1)$: $3^{-1} = \frac{1}{3} < (-1)^2 = 1$ is true

$P(-2)$: $3^{-2} = \frac{1}{9} < (-2)^2 = 4$ is true

$P(-3)$: $3^{-3} = \frac{1}{27} < (-3)^2 = 9$ is true

Thuse $\forall n \in A\ P(n)$ is true,

QUESTION 4: Use mathematical induction to prove that for every positive integer n,

$$2 + 2(2^2) + 3(2^3) + \cdots + n2^n = (n-1)2^{n+1} + 2$$

$\frac{4}{4}$

Let $P(n)$: $2 + 2(2^2) + \cdots + n2^n = (n-1)\,2^{n+1} + 2$, $n \geq 1$

basis step: we need to prove $P(1)$ is true

$P(1)$: $2 = (1-1)\,2^{1+1} + 2$ is clearly true.

Inductive step: we want to show that $\forall k\,(P(k) \to P(k+1))$ is true. Suppose $P(k)$ is true for some integer $k \geq 1$ i.e:

$$\sum_{i=1}^{k} i2^i = (k-1)\,2^{k+1} + 2$$

we want to show $P(k+1)$ is true i.e $\sum_{i=1}^{k+1} i2^i = k\,2^{k+2} + 2$

$$\sum_{i=1}^{k+1} i2^i = \sum_{i=1}^{k} i2^i + (k+1)\,2^{k+1}$$

$$= (k-1)\,2^{k+1} + 2 + (k+1)\,2^{k+1}$$

$$= 2^{k+1}\,[k-1+k+1] + 2$$

$$= 2^{k+1}\,(2k) + 2$$

$$= k\,2^{k+2} + 2$$

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so $P(k+1)$ is true.

Using mathematical induction, $P(n)$ is true for all $n \geq 1$.