Moment Generating Functions

Given a random variable X, its moment generating function (mgf) is given by:

$$m_X(t) = E_X\left[e^{tX}\right]$$

Ex. 1a) Let $X \sim \exp(\lambda)$

$$m_X(t) = \int_0^\infty e^{tx}\lambda e^{-\lambda x}\,dx = \int_0^\infty \lambda e^{(t-\lambda)x}\,dx$$

$$= \frac{\lambda}{\lambda - t}\underbrace{\int_0^\infty (\lambda - t)e^{-(\lambda - t)x}\,dx}_{=1} = \frac{\lambda}{\lambda - t} \text{ for } t < \lambda$$

Properties of the mgf:

i) $m_X(0) = E_X\left[e^{0X}\right] = E_X[1] = 1$

ii) $m_X^{(k)}(0) = E_X\left[X^k\right]$ where $m_X^{(k)}(t) = \frac{d^k}{dt^k} m_X(t)$

pf.

$$\frac{d^k}{dt^k} E_X\left[e^{tX}\right] \overset{*}{=} E_X\left[\frac{d^k}{dt^k} e^{tX}\right] = E_X\left[\frac{d^{k-1}}{dt^{k-1}}\left(Xe^{tX}\right)\right] = \cdots$$

$$= E_X\left[X^k e^{tX}\right]$$

so $m_X^{(k)}(0) = E_X\left[X^k e^0\right] = E_X\left[X^k\right]$

Ex. 1b) Let $X \sim \exp(\lambda)$ so $m_X(t) = \frac{\lambda}{\lambda - t}$ for $t < \lambda$, then

$$m_X'(t) = \frac{(\lambda - t)\cdot 0 - \lambda(-1)}{(\lambda - t)^2} = \frac{\lambda}{(\lambda - t)^2} \Rightarrow m_X'(0) = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}$$

$$m_X''(t) = \frac{(\lambda - t)^2\cdot 0 - \lambda[-2(\lambda - t)]}{(\lambda - t)^4} = \frac{2\lambda(\lambda - t)}{(\lambda - t)^4} \Rightarrow m_X''(0) = \frac{2\lambda^2}{\lambda^4} = \frac{2}{\lambda^2}$$

Note:

i) Not all random variables have mgfs.

- An example is the Cauchy Distribution. its pdf is given by

$$f_x(x;\nu,\tau) = \frac{1}{\pi\tau\left(1+\left(\frac{x-\nu}{\tau}\right)^2\right)} = \frac{1}{\pi\tau}\left(\frac{\tau^2}{(x-\nu)^2+\tau^2}\right)$$

ii) If two distributions have the same mgf, then they are identically distributed.

- Consequently, the mgf is another way to specify the probability distribution.
- The proof of (ii) relies on the theory of Laplace Transform.

Result: Suppose $X_1, X_2 \dots X_n$ are i.i.d (independent & identically distributed). Define the r.v. S as,

$$S = \sum_{i=1}^{n} (a_i X_i + b_i) \quad \text{where } a_i, b_i \in \mathbb{R}$$

then it follows $m_S(t) = \exp\left(t\sum_{i=1}^{n} b_i\right)\prod_{i=1}^{n} m_{X_i}(a_i t)$

Pf:

$$m_S(t) = E[e^{tS}] = E\left[\exp\left(t\sum_{i=1}^{n}(a_i X_i + b_i)\right)\right]$$

$$= \exp\left(t\sum_{i=1}^{n} b_i\right) E\left[\prod_{i=1}^{n} e^{t a_i X_i}\right] \underset{\text{by independence}}{=} \exp\left(t\sum_{i=1}^{n} b_i\right)\prod_{i=1}^{n} E_{X_i}\left[e^{t a_i X_i}\right]$$

$$= \exp\left(t\sum_{i=1}^{n} b_i\right)\prod_{i=1}^{n} m_{X_i}(a_i t) \qquad \text{Q.E.D}$$

Corollary I: Suppose $Y = aX + b$, if X has a mgf then it follows

$$m_Y(t) = e^{b} m_X(at)$$

Corollary II: Suppose $X_1, X_2 \ldots X_n$ are iid. Define the r.v. S as,

$$S = \sum_{i=1}^{n} X_i$$

Then $m_S(t) = \left(m_X(t)\right)^n$.

Ex. 2) Let $Y \sim \text{Gamma}(K, \lambda)$, then

$$m_Y(t) = \frac{\lambda^K}{\Gamma(K)} \int_0^\infty e^{ty} y^{K-1} e^{-\lambda y}\, dy = \frac{\lambda^K}{\Gamma(K)} \int_0^\infty y^{K-1} e^{(t-\lambda)y}\, dy$$

$$= \frac{\lambda^K}{(\lambda - t)^K} \underbrace{\frac{(\lambda - t)^K}{\Gamma(K)} \int_0^\infty y^{K-1} e^{-(\lambda - t)y}\, dy}_{=1} = \left(\frac{\lambda}{\lambda - t}\right)^K \text{ for } t < \lambda$$

But, $\frac{\lambda}{\lambda - t}$ is the mgf of an $\exp(\lambda)$, thus if $X_1, X_2, \ldots X_n$ are iid $\exp(\lambda)$, then $S = \sum_{i=1}^{n} X_i \sim \text{Gamma}(n, \lambda)$