

**Exercise 1 :**

$$\text{a) } \int \frac{dx}{e^{-x} \sqrt{e^{2x} - 1}} \stackrel{t=e^x}{=} \int \frac{dt}{\sqrt{t^2 - 1}} = \cosh^{-1}(e^x) + c.$$

b)

$$\begin{aligned} \int \frac{\tan x}{\sqrt{4 - \cos^4 x}} dx &\stackrel{t=\frac{1}{2} \cos^2(x)}{=} -\frac{1}{4} \int \frac{dt}{t\sqrt{1-t^2}} \\ &= \frac{1}{4} \operatorname{sech}^{-1}\left(\frac{1}{2} \cos^2(x)\right) + c. \end{aligned}$$

c)

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{1}{x} - \frac{1}{\sin x} &= \lim_{x \rightarrow 0} \frac{\sin x - x}{x \sin x} \\ &= \lim_{x \rightarrow 0} \frac{\cos x - 1}{\sin x + x \cos x} \\ &= \lim_{x \rightarrow 0} \frac{-\sin x}{2 \cos x - x \sin x} = 0. \end{aligned}$$

**Exercise 2 :**

a)

$$\begin{aligned} \int e^{2x} \sin x dx &\stackrel{u=e^{2x}, v'=\sin x}{=} -e^{2x} \cos x + 2 \int e^{2x} \cos x dx \\ &\stackrel{u=e^{2x}, v'=\cos x}{=} -e^{2x} \cos x + 2e^{2x} \sin x - 4 \int e^{2x} \sin x dx. \end{aligned}$$

$$\text{Then } \int e^{2x} \sin x dx = -\frac{1}{5} e^{2x} \cos x + \frac{2}{5} e^{2x} \sin x + c.$$

b)

$$\begin{aligned} \int \sec^4 x \tan^7 x dx &\stackrel{u=\tan x}{=} \int u^7 (1+u^2) du \\ &= \frac{1}{8} \tan^8 x + \frac{1}{10} \tan^{10} x + c. \end{aligned}$$

c)

$$\begin{aligned} \int \frac{dx}{x^3 \sqrt{x^2 - 4}} &\stackrel{x=2 \sec \theta}{=} \frac{1}{8} \int \cos^2 \theta d\theta \\ &= \frac{1}{16} \int (1 + \cos(2\theta)) d\theta = \frac{1}{16} (\theta + \cos \theta \sin \theta) + c \\ &= \frac{1}{16} \left( \sec^{-1} \left( \frac{x}{2} \right) + \frac{2\sqrt{x^2 - 4}}{x^2} \right) + c \end{aligned}$$

Or

$$\begin{aligned} \int \frac{dx}{x^3 \sqrt{x^2 - 4}} &\stackrel{x^2 - 4 = t^2}{=} \int \frac{dt}{(t^2 + 4)^2} \\ &= \frac{1}{16} \tan^{-1} \left( \frac{t}{2} \right) + \frac{1}{16} \ln \left( \frac{t}{2} \right) + c \\ &= \frac{1}{16} \tan^{-1} \left( \frac{\sqrt{x^2 - 4}}{2} \right) + \frac{1}{16} \ln \left( \frac{\sqrt{x^2 - 4}}{2} \right) + c \end{aligned}$$

**Exercise 3 :**

a)

$$\begin{aligned} \int \frac{6x^2 + x + 8}{x^3 + 4x} dx &= \int \left( \frac{2}{x} + \frac{4x + 1}{x^2 + 4} \right) dx \\ &= 2 \ln |x| + 2 \ln(x^2 + 4) + \frac{1}{2} \tan^{-1} \left( \frac{x}{2} \right) + c. \end{aligned}$$

b)

$$\begin{aligned} \int \frac{dx}{(x+1)^{\frac{5}{6}} - (x+1)^{\frac{1}{2}}} &\stackrel{x+1=t^6}{=} \int \frac{6t^5}{t^5 - t^3} dt \\ &= 6 \int \frac{t^2}{t^2 - 1} dt = 6t - 3 \ln \left| \frac{1+t}{1-t} \right| + c \\ &= 6(x+1)^{\frac{1}{6}} - 3 \ln \left| \frac{1 + (x+1)^{\frac{1}{6}}}{1 - (x+1)^{\frac{1}{6}}} \right| + c \end{aligned}$$

c)  $\int \frac{dx}{2 + \cos x} \stackrel{t = \tan(\frac{x}{2})}{=} \int \frac{2dt}{3 + t^2} dt = \frac{2}{\sqrt{3}} \tan^{-1} \left( \frac{1}{\sqrt{3}} \tan \left( \frac{x}{2} \right) \right) + c$