Chapter 2
Matrices

2.1 Operations with Matrices

- Matrix:

$$A=[a_{ij}]=\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n} \in M_{m\times n}$$

(i, j)-th entry: a_{ij}

row: m

column: n

size: $m\times n$

- *i*-th row vector

$$r_i = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix}$$

row matrix

- *j*-th column vector

$$c_j = \begin{bmatrix} c_{1j} \\ c_{2j} \\ \vdots \\ c_{mj} \end{bmatrix}$$

column matrix

- Square matrix: $m = n$

- Diagonal matrix:

$$A = diag(d_1, d_2, \cdots, d_n) = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} \in M_{n\times n}$$

- Trace:

If $A = [a_{ij}]_{n\times n}$

Then $Tr(A) = a_{11} + a_{22} + \cdots + a_{nn}$

- Ex:

$$A=\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix}=\begin{bmatrix}r_1\\ r_2\end{bmatrix}$$

$$\Rightarrow \quad r_1=\begin{bmatrix}1 & 2 & 3\end{bmatrix},\ \ r_2=\begin{bmatrix}4 & 5 & 6\end{bmatrix}$$

$$A=\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix}=\begin{bmatrix}c_1 & c_2 & c_3\end{bmatrix}$$

$$\Rightarrow \quad c_1=\begin{bmatrix}1\\ 4\end{bmatrix},\ c_2=\begin{bmatrix}2\\ 5\end{bmatrix},\ \ c_3=\begin{bmatrix}3\\ 6\end{bmatrix}$$

- Equal matrix:

If $A=[a_{ij}]_{m\times n}$, $B=[b_{ij}]_{m\times n}$

Then $A=B$ if and only if $a_{ij}=b_{ij}\ \forall 1\le i\le m,\ 1\le j\le n$

- Ex 1: (Equal matrix)

$$A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix} \qquad B=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$$

If $A=B$

Then $a=1,\ b=2,\ c=3,\ d=4$

- Matrix addition:

$$\text{If } A=[a_{ij}]_{m\times n},\ B=[b_{ij}]_{m\times n}$$

$$\text{Then } A+B=[a_{ij}]_{m\times n}+[b_{ij}]_{m\times n}=[a_{ij}+b_{ij}]_{m\times n}$$

- Ex 2: (Matrix addition)

$$\begin{bmatrix}-1 & 2\\ 0 & 1\end{bmatrix}+\begin{bmatrix}1 & 3\\ -1 & 2\end{bmatrix}=\begin{bmatrix}-1+1 & 2+3\\ 0-1 & 1+2\end{bmatrix}=\begin{bmatrix}0 & 5\\ -1 & 3\end{bmatrix}$$

$$\begin{bmatrix}1\\ -3\\ -2\end{bmatrix}+\begin{bmatrix}-1\\ 3\\ 2\end{bmatrix}=\begin{bmatrix}1-1\\ -3+3\\ -2+2\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

- Scalar multiplication:

If $A=[a_{ij}]_{m\times n}$, c : scalar

Then $cA=[ca_{ij}]_{m\times n}$

- Matrix subtraction:

$A-B=A+(-1)B$

- Ex 3: (Scalar multiplication and matrix subtraction)

$$A=\begin{bmatrix} 1 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix} \qquad B=\begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix}$$

Find (*a*) 3*A*, (*b*) –*B*, (*c*) 3*A* – *B*

Sol:

(a)

$$3A = 3\begin{bmatrix} 1 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3(1) & 3(2) & 3(4) \\ 3(-3) & 3(0) & 3(-1) \\ 3(2) & 3(1) & 3(2) \end{bmatrix} = \begin{bmatrix} 3 & 6 & 12 \\ -9 & 0 & -3 \\ 6 & 3 & 6 \end{bmatrix}$$

(b)

$$-B = (-1)\begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ -1 & 4 & -3 \\ 1 & -3 & -2 \end{bmatrix}$$

(c)

$$3A - B = \begin{bmatrix} 3 & 6 & 12 \\ -9 & 0 & -3 \\ 6 & 3 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 12 \\ -10 & 4 & -6 \\ 7 & 0 & 4 \end{bmatrix}$$

- Matrix multiplication:

If $A = [a_{ij}]_{m\times n},\ B = [b_{ij}]_{n\times p}$

Then $AB = [a_{ij}]_{m\times n}[b_{ij}]_{n\times p} = [c_{ij}]_{m\times p}$

Size of AB

where $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & \vdots & b_{2j} & \cdots & b_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ b_{n1} & \cdots & b_{nj} & \cdots & b_{nn} \end{bmatrix} = \begin{bmatrix} \\ c_{i1} & c_{i2} & \cdots & c_{ij} & \cdots & c_{in} \\ \\ \end{bmatrix}$$

- Notes: (1) $A+B = B+A$, (2) $AB \neq BA$

- Ex 4: (Find AB)

$$A=\begin{bmatrix}-1 & 3\\ 4 & -2\\ 5 & 0\end{bmatrix} \qquad B=\begin{bmatrix}-3 & 2\\ -4 & 1\end{bmatrix}$$

Sol:

$$AB=\begin{bmatrix}(-1)(-3)+(3)(-4) & (-1)(2)+(3)(1)\\ (4)(-3)+(-2)(-4) & (4)(2)+(-2)(1)\\ (5)(-3)+(0)(-4) & (5)(2)+(0)(1)\end{bmatrix}$$

$$=\begin{bmatrix}-9 & 1\\ -4 & 6\\ -15 & 10\end{bmatrix}$$

- Matrix form of a system of linear equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

m linear equations

$$\Downarrow$$

$$\underset{A}{\underset{\shortparallel}{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}} \underset{x}{\underset{\shortparallel}{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}} = \underset{b}{\underset{\shortparallel}{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}}$$

Single matrix equation

$$\underset{m\times n}{A}\,\underset{n\times 1}{x} = \underset{m\times 1}{b}$$

- Partitioned matrices:

submatrix

$$A=\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right]=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$A=\left[\begin{array}{cccc} a_{11} & a_{12} & a_{13} & a_{14} \\ \hline a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right]=\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$$

$$A=\left[\begin{array}{c|c|c|c} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right]=\begin{bmatrix} c_1 & c_2 & c_3 & c_4 \end{bmatrix}$$

Keywords in Section 2.1:

- row vector:
- column vector:
- diagonal matrix:
- trace:
- equality of matrices:
- matrix addition:
- scalar multiplication:
- matrix multiplication:
- partitioned matrix:

2.2 Properties of Matrix Operations

- Three basic matrix operators:

  (1) matrix addition

  (2) scalar multiplication

  (3) matrix multiplication

- Zero matrix: $0_{m\times n}$

- Identity matrix of order *n*: I_n

- Properties of matrix addition and scalar multiplication:

If $A, B, C \in M_{m\times n}$, c, d : scalar

Then (1) $A+B = B + A$

(2) $A + (B + C) = (A + B) + C$

(3) $(cd) A = c (dA)$

(4) $1A = A$

(5) $c(A+B) = cA + cB$

(6) $(c+d) A = cA + dA$

- Properties of zero matrices:

If $A \in M_{m\times n}$, $\quad c : scalar$

Then (1) $A + 0_{m\times n} = A$

(2) $A + (-A) = 0_{m\times n}$

(3) $cA = 0_{m\times n} \Rightarrow c = 0 \ or \ A = 0_{m\times n}$

- Notes:

(1) $0_{m\times n}$: **the additive identity** for the set of all $m\times n$ matrices

(2) $-A$: **the additive inverse** of A

- Transpose of a matrix:

$$\text{If}\quad A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \in M_{m\times n}$$

$$\text{Then}\quad A^T=\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix} \in M_{n\times m}$$

- Ex 8: (Find the transpose of the following matrix)

$$(a)\quad A=\begin{bmatrix}2\\8\end{bmatrix}\qquad (b)\quad A=\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}\qquad (c)\quad A=\begin{bmatrix}0&1\\2&4\\1&-1\end{bmatrix}$$

Sol: (a)

$$A=\begin{bmatrix}2\\8\end{bmatrix}\qquad \Rightarrow A^T=\begin{bmatrix}2&8\end{bmatrix}$$

(b)

$$A=\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}\Rightarrow A^T=\begin{bmatrix}1&4&7\\2&5&8\\3&6&9\end{bmatrix}$$

(c)

$$A=\begin{bmatrix}0&1\\2&4\\1&-1\end{bmatrix}\quad\Rightarrow A^T=\begin{bmatrix}0&2&1\\1&4&-1\end{bmatrix}$$

- Properties of transposes:

(1) $(A^T)^T = A$

(2) $(A+B)^T = A^T + B^T$

(3) $(cA)^T = c(A^T)$

(4) $(AB)^T = B^T A^T$

- Symmetric matrix:

  A square matrix A is **symmetric** if $A = A^T$

- Skew-symmetric matrix:

  A square matrix A is **skew-symmetric** if $A^T = -A$

- Ex:

  If $A = \begin{bmatrix} 1 & 2 & 3 \\ a & 4 & 5 \\ b & c & 6 \end{bmatrix}$ is symmetric, find a, b, c?

Sol:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ a & 4 & 5 \\ b & c & 6 \end{bmatrix} \quad A^T = \begin{bmatrix} 1 & a & b \\ 2 & 4 & c \\ 3 & 5 & 6 \end{bmatrix} \quad \begin{aligned} & A = A^T \\ & \Rightarrow a = 2, b = 3, c = 5 \end{aligned}$$

- Ex:

$$\text{If } A=\begin{bmatrix}0 & 1 & 2\\ a & 0 & 3\\ b & c & 0\end{bmatrix} \text{ is a skew-symmetric, find } a, b, c?$$

Sol:

$$A=\begin{bmatrix}0 & 1 & 2\\ a & 0 & 3\\ b & c & 0\end{bmatrix} \qquad -A^T=\begin{bmatrix}0 & -a & -b\\ -1 & 0 & -c\\ -2 & -3 & 0\end{bmatrix}$$

$$A=-A^T \Rightarrow a=-1, b=-2, c=-3$$

- Note: AA^T is symmetric

Pf: $$(AA^T)^T=(A^T)^T A^T = AA^T$$

$\therefore AA^T$ is symmetric

- Real number:

  $ab = ba$ (Commutative law for multiplication)

- Matrix:

  $\underset{m\times n}{A}\underset{n\times p}{B} \neq BA$

  Three situations:

  (1) If $m \neq p$, then AB is defined, BA is undefined.

  (2) If $m = p, m \neq n$, then $AB \in M_{m\times m}$, $BA \in M_{n\times n}$ (Sizes are not the same)

  (3) If $m = p = n$, then $AB \in M_{m\times m}$, $BA \in M_{m\times m}$

  (Sizes are the same, but matrices are not equal)

- Ex 4:

Sow that AB and BA are not equal for the matrices.

$$A=\begin{bmatrix}1 & 3\\ 2 & -1\end{bmatrix} \quad \text{and} \quad B=\begin{bmatrix}2 & -1\\ 0 & 2\end{bmatrix}$$

Sol:

$$AB=\begin{bmatrix}1 & 3\\ 2 & -1\end{bmatrix}\begin{bmatrix}2 & -1\\ 0 & 2\end{bmatrix}=\begin{bmatrix}2 & 5\\ 4 & -4\end{bmatrix}$$

$$BA=\begin{bmatrix}2 & -1\\ 0 & 2\end{bmatrix}\begin{bmatrix}1 & 3\\ 2 & -1\end{bmatrix}=\begin{bmatrix}0 & 7\\ 4 & -2\end{bmatrix}$$

- Note: $AB \neq BA$

- Real number:

$$ac = bc,\ c \neq 0$$

$$\Rightarrow\ a = b$$ (Cancellation law)

- Matrix:

$$AC = BC \qquad C \neq 0$$

(1) If C is invertible, then $A = B$

(2) If C is not invertible, then $A \neq B$ (Cancellation is not valid)

- Ex 5: (An example in which cancellation is not valid)

Show that $AC=BC$

$$A=\begin{bmatrix}1 & 3\\ 0 & 1\end{bmatrix},\quad B=\begin{bmatrix}2 & 4\\ 2 & 3\end{bmatrix},\quad C=\begin{bmatrix}1 & -2\\ -1 & 2\end{bmatrix}$$

Sol:

$$AC=\begin{bmatrix}1 & 3\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & -2\\ -1 & 2\end{bmatrix}=\begin{bmatrix}-2 & 4\\ -1 & 2\end{bmatrix}$$

$$BC=\begin{bmatrix}2 & 4\\ 2 & 3\end{bmatrix}\begin{bmatrix}1 & -2\\ -1 & 2\end{bmatrix}=\begin{bmatrix}-2 & 4\\ -1 & 2\end{bmatrix}$$

So $AC=BC$

But $A\neq B$

Keywords in Section 2.2:

- zero matrix:
- identity matrix:
- transpose matrix:
- symmetric matrix:
- skew-symmetric matrix:

2.3 The Inverse of a Matrix

- Inverse matrix:

Consider $A \in M_{n \times n}$

If there exists a matrix $B \in M_{n \times n}$ such that $AB = BA = I_n$,

Then (1) A is **invertible** (or **nonsingular**)

(2) B is **the inverse** of A

- Note:

A matrix that does not have an inverse is called **noninvertible** (or **singular**).

- Thm 2.7: (The inverse of a matrix is unique)

If B and C are both inverses of the matrix A, then $B = C$.

Pf:
$$AB = I$$
$$C(AB) = CI$$
$$(CA)B = C$$
$$IB = C$$
$$B = C$$

Consequently, the inverse of a matrix is unique.

- Notes:

(1) The inverse of A is denoted by A^{-1}

(2) $AA^{-1} = A^{-1}A = I$

- Find the inverse of a matrix by Gauss-Jordan Elimination:

$$[A \;|\; I] \xrightarrow{\text{Gauss-Jordan Elimination}} [I \;|\; A^{-1}]$$

- Ex 2: (Find the inverse of the matrix)

$$A = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}$$

Sol:

$$AX = I$$

$$\begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x_{11} + 4x_{21} & x_{12} + 4x_{22} \\ -x_{11} - 3x_{21} & -x_{12} - 3x_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{array}{rcrcl} x_{11} & + & 4x_{21} & = & 1 \\ -x_{11} & - & 3x_{21} & = & 0 \end{array} \qquad (1)$$

$$\begin{array}{rcrcl} x_{12} & + & 4x_{22} & = & 0 \\ -x_{12} & - & 3x_{22} & = & 1 \end{array} \qquad (2)$$

$$(1) \Rightarrow \left[\begin{array}{cc:c} 1 & 4 & 1 \\ -1 & -3 & 0 \end{array}\right] \xrightarrow{r_{12}^{(1)},\ r_{21}^{(-4)}} \left[\begin{array}{cc:c} 1 & 0 & -3 \\ 0 & 1 & 1 \end{array}\right] \Rightarrow x_{11} = -3, x_{21} = 1$$

$$(2) \Rightarrow \left[\begin{array}{cc:c} 1 & 4 & 0 \\ -1 & -3 & 1 \end{array}\right] \xrightarrow{r_{12}^{(1)},\ r_{21}^{(-4)}} \left[\begin{array}{cc:c} 1 & 0 & -4 \\ 0 & 1 & 1 \end{array}\right] \Rightarrow x_{12} = -4, x_{22} = 1$$

Thus

$$X = A^{-1} = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix} \ (AX = I = AA^{-1})$$

- Note:

$$\begin{bmatrix} 1 & 4 & \vdots & 1 & 0 \\ -1 & -3 & \vdots & 0 & 1 \end{bmatrix} \xrightarrow[r_{12}^{(1)},\, r_{21}^{(-4)}]{\text{Gauss-Jordan Elimination}} \begin{bmatrix} 1 & 0 & \vdots & -3 & -4 \\ 0 & 1 & \vdots & 1 & 1 \end{bmatrix}$$

$$\quad A \qquad\qquad I \qquad\qquad\qquad\qquad I \qquad\qquad A^{-1}$$

If A can't be row reduced to I, then A is singular.

- Ex 3: (Find the inverse of the following matrix)

$$A=\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix}$$

Sol:

$$[A \vdots I]=\left[\begin{array}{rrr|rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ -6 & 2 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{r_{12}^{(-1)}}\left[\begin{array}{rrr|rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ -6 & 2 & 3 & 0 & 0 & 1 \end{array}\right]\xrightarrow{r_{13}^{(6)}}\left[\begin{array}{rrr|rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & -4 & 3 & 6 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{r_{23}^{(4)}}\left[\begin{array}{rrr|rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 2 & 4 & 1 \end{array}\right]\xrightarrow{r_{3}^{(-1)}}\left[\begin{array}{rrr|rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right]$$

$$\xrightarrow{r_{32}^{(1)}} \left[\begin{array}{ccc:ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right] \xrightarrow{r_{21}^{(1)}} \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & -2 & -3 & -1 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -1 & -4 & -1 \end{array}\right]$$

$$= [\, I \;\vdots\; A^{-1} \,]$$

So the matrix A is invertible, and its inverse is

$$A^{-1} = \begin{bmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{bmatrix}$$

- Check:

$$AA^{-1} = A^{-1}A = I$$

- Power of a square matrix:

$$(1) A^0 = I$$

$$(2) A^k = \underbrace{AA\cdots A}_{k \text{ factors}} \qquad (k > 0)$$

$$(3) A^r \cdot A^s = A^{r+s} \qquad r, s: \text{integers}$$

$$(A^r)^s = A^{rs}$$

$$(4) D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} \Rightarrow D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix}$$

- Thm 2.8： (Properties of inverse matrices)

If A is an invertible matrix, k is a positive integer, and c is a scalar not equal to zero, then

(1) A^{-1} is invertible and $(A^{-1})^{-1}=A$

(2) A^{k} is invertible and $(A^{k})^{-1}=\underbrace{A^{-1}A^{-1}\cdots A^{-1}}_{k \text{ factors}}=(A^{-1})^{k}=A^{-k}$

(3) cA is invertible and $(cA)^{-1}=\frac{1}{c}A^{-1}, c\neq 0$

(4) A^{T} is invertible and $(A^{T})^{-1}=(A^{-1})^{T}$

- Thm 2.9: (The inverse of a product)

If A and B are invertible matrices of size n, then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}$$

Pf:

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = A(I)A^{-1} = (AI)A^{-1} = AA^{-1} = I$$

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}(I)B = B^{-1}(IB) = B^{-1}B = I$$

If AB is invertible, then its inverse is unique.

So $(AB)^{-1} = B^{-1}A^{-1}$

- Note:

$$(A_1A_2A_3\cdots A_n)^{-1} = A_n^{-1}\cdots A_3^{-1}A_2^{-1}A_1^{-1}$$

- Thm 2.10 (Cancellation properties)

If C is an invertible matrix, then the following properties hold:

(1) If $AC=BC$, then $A=B$ (Right cancellation property)

(2) If $CA=CB$, then $A=B$ (Left cancellation property)

Pf:

$$AC = BC$$

$$(AC)C^{-1} = (BC)C^{-1} \qquad \text{(C is invertible, so } C^{-1} \text{ exists)}$$

$$A(CC^{-1}) = B(CC^{-1})$$

$$AI = BI$$

$$A = B$$

- Note:

If C is not invertible, then cancellation is not valid.

- Thm 2.11: (Systems of equations with unique solutions)

If A is an invertible matrix, then the system of linear equations $Ax = b$ has a unique solution given by

$$x = A^{-1}b$$

Pf:

$$Ax = b$$

$$A^{-1}Ax = A^{-1}b \qquad (A \text{ is nonsingular})$$

$$Ix = A^{-1}b$$

$$x = A^{-1}b$$

If x_1 and x_2 were two solutions of equation $Ax = b$.

then $Ax_1 = b = Ax_2 \Rightarrow x_1 = x_2$ (Left cancellation property)

This solution is unique.

- Note:

For square systems (those having the same number of equations as variables), Theorem 2.11 can be used to determine whether the system has a unique solution.

- Note:

$Ax = b$ (A is an invertible matrix)

$$\left[A \mid b\right] \xrightarrow{A^{-1}} \left[A^{-1}A \mid A^{-1}b\right] = \left[I \mid A^{-1}b\right]$$

Keywords in Section 2.3:

- inverse matrix:
- invertible:
- nonsingular:
- singular:
- power:

2.4 Elementary Matrices

- Row elementary matrix:

  An $n \times n$ matrix is called an elementary matrix if it can be obtained from the identity matrix I_n by a single elementary operation.

- Three row elementary matrices:

  (1) $R_{ij} = r_{ij}(I)$ Interchange two rows.

  (2) $R_i^{(k)} = r_i^{(k)}(I)$ $(k \neq 0)$ Multiply a row by a nonzero constant.

  (3) $R_{ij}^{(k)} = r_{ij}^{(k)}(I)$ Add a multiple of a row to another row.

- Note:

  Only do a single elementary row operation.

- Ex 1: (Elementary matrices and nonelementary matrices)

$$(a)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Yes $(r_2^{(3)}(I_3))$

$$(b)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

No (not square)

$$(c)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

No (Row multiplication must be by a nonzero constant)

$$(d)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Yes $(r_{23}(I_3))$

$$(e)\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

Yes $(r_{12}^{(2)}(I_2))$

$$(f)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

No (Use two elementary row operations)

- Thm 2.12: (Representing elementary row operations)

Let E be the elementary matrix obtained by performing an elementary row operation on I_m. If that same elementary row operation is performed on an $m \times n$ matrix A, then the resulting matrix is given by the product EA.

$$r(I) = E$$
$$r(A) = EA$$

- Notes:

(1) $r_{ij}(A) = R_{ij}A$

(2) $r_i^{(k)}(A) = R_i^{(k)}A$

(3) $r_{ij}^{(k)}(A) = R_{ij}^{(k)}A$

- Ex 2: (Elementary matrices and elementary row operation)

$$(a)\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}0&2&1\\1&-3&6\\3&2&-1\end{bmatrix}=\begin{bmatrix}1&-3&6\\0&2&1\\3&2&-1\end{bmatrix}(r_{12}(A)=R_{12}A)$$

$$(b)\begin{bmatrix}1&0&0\\0&\frac{1}{2}&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&-4&1\\0&2&6&-4\\0&1&3&1\end{bmatrix}=\begin{bmatrix}1&0&-4&1\\0&1&3&-2\\0&1&3&1\end{bmatrix}(r_2^{(\frac{1}{2})}(A)=R_2^{(\frac{1}{2})}A)$$

$$(c)\begin{bmatrix}1&0&0\\2&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&-1\\-2&-2&3\\0&4&5\end{bmatrix}=\begin{bmatrix}1&0&-1\\0&-2&1\\0&4&5\end{bmatrix}(r_{12}^{(2)}(A)=R_{12}^{(5)}A)$$

- Ex 3: (Using elementary matrices)

  Find a sequence of elementary matrices that can be used to write the matrix A in row-echelon form.

$$A = \begin{bmatrix} 0 & 1 & 3 & 5 \\ 1 & -3 & 0 & 2 \\ 2 & -6 & 2 & 0 \end{bmatrix}$$

Sol:

$$E_1 = r_{12}(I_3) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_2 = r_{13}^{(-2)}(I_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$$E_3 = r_3^{(\frac{1}{2})}(I_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$$A_1 = r_{12}(A) = E_1 A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 3 & 5 \\ 1 & -3 & 0 & 2 \\ 2 & -6 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 2 & -6 & 2 & 0 \end{bmatrix}$$

$$A_2 = r_{13}^{(-2)}(A_1) = E_2 A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 2 & -6 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & -4 \end{bmatrix}$$

$$A_3 = r_3^{(\frac{1}{2})}(A_2) = E_3 A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & -4 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & -2 \end{bmatrix} = B$$

row-echelon form

$$\therefore\ B = E_3 E_2 E_1 A \quad \text{or} \quad B = r_3^{(\frac{1}{2})}(r_{13}^{(-2)}(r_{12}(A)))$$

- Row-equivalent:

  Matrix B is **row-equivalent** to A if there exists a finite number of elementary matrices such that

$$B = E_k E_{k-1} \cdots E_2 E_1 A$$

- Thm 2.13: (Elementary matrices are invertible)

  If E is an elementary matrix, then E^{-1} exists and is an elementary matrix.

- Notes:

  (1) $(R_{ij})^{-1} = R_{ij}$

  (2) $(R_i^{(k)})^{-1} = R_i^{(\frac{1}{k})}$

  (3) $(R_{ij}^{(k)})^{-1} = R_{ij}^{(-k)}$

- Ex:

Elementary Matrix — Inverse Matrix

$$E_1=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}=R_{12} \qquad (R_{12})^{-1}=E_1^{-1}=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}=R_{12}$$ (Elementary Matrix)

$$E_2=\begin{bmatrix}1&0&0\\0&1&0\\-2&0&1\end{bmatrix}=R_{13}^{(-2)} \qquad (R_{13}^{(-2)})^{-1}=E_2^{-1}=\begin{bmatrix}1&0&0\\0&1&0\\2&0&1\end{bmatrix}=R_{13}^{(2)}$$ (Elementary Matrix)

$$E_3=\begin{bmatrix}1&0&0\\0&1&0\\0&0&\frac{1}{2}\end{bmatrix}=R_3^{(\frac{1}{2})} \qquad (R_3^{(\frac{1}{2})})^{-1}=E_3^{-1}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&2\end{bmatrix}=R_3^{(2)}$$ (Elementary Matrix)

- Thm 2.14: (A property of invertible matrices)

A square matrix A is invertible if and only if it can be written as the product of elementary matrices.

Pf: (1) Assume that A is the product of elementary matrices.

(a) Every elementary matrix is invertible.

(b) The product of invertible matrices is invertible.

Thus A is invertible.

(2) If A is invertible, $A\mathbf{x}=0$ has only the trivial solution. (Thm. 2.11)

$$\Rightarrow [A \vdots 0] \rightarrow [I \vdots 0]$$

$$\Rightarrow E_k \cdots E_3 E_2 E_1 A = I$$

$$\Rightarrow A = E_1^{-1} E_2^{-1} E_3^{-1} \cdots E_k^{-1}$$

Thus A can be written as the product of elementary matrices.

- Ex 4:

Find a sequence of elementary matrices whose product is

$$A=\begin{bmatrix}-1 & -2\\ 3 & 8\end{bmatrix}$$

Sol:

$$A=\begin{bmatrix}-1 & -2\\ 3 & 8\end{bmatrix}\xrightarrow{r_1^{(-1)}}\begin{bmatrix}1 & 2\\ 3 & 8\end{bmatrix}\xrightarrow{r_{12}^{(-3)}}\begin{bmatrix}1 & 2\\ 0 & 2\end{bmatrix}$$

$$\xrightarrow{r_2^{(\frac{1}{2})}}\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}\xrightarrow{r_{21}^{(-2)}}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=I$$

Therefore $R_{21}^{(-2)}R_2^{(\frac{1}{2})}R_{12}^{(-3)}R_1^{(-1)}A=I$

Thus $A=(R_1^{(-1)})^{-1}(R_{12}^{(-3)})^{-1}(R_2^{(\frac{1}{2})})^{-1}(R_{21}^{(-2)})^{-1}$

$$=R_1^{(-1)}R_{12}^{(3)}R_2^{(2)}R_{21}^{(2)}$$

$$=\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 3 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}$$

- Note:

If A is invertible

Then $E_k \cdots E_3 E_2 E_1 A = I$

$$A^{-1} = E_k \cdots E_3 E_2 E_1$$

$$E_k \cdots E_3 E_2 E_1 [A \vdots I] = [I \vdots A^{-1}]$$

- Thm 2.15: (Equivalent conditions)

If A is an $n \times n$ matrix, then the following statements are equivalent.

(1) A is invertible.

(2) $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $n \times 1$ column matrix $\mathbf{b}$.

(3) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(4) A is row-equivalent to I_n .

(5) A can be written as the product of elementary matrices.

- *LU*-factorization:

If the $n \times n$ matrix A can be written as the product of a lower triangular matrix L and an upper triangular matrix U, then A=LU is an LU-factorization of A

$$A = LU$$

L is a lower triangular matrix

U is an upper triangular matrix

- Note:

If a square matrix A can be row reduced to an upper triangular matrix U using only the row operation of adding a multiple of one row to another row below it, then it is easy to find an LU-factorization of A.

$$E_k \cdots E_2 E_1 A = U$$

$$A = E_1^{-1} E_2^{-1} \cdots E_k^{-1} U$$

$$A = LU$$

- Ex 5: (*LU*-factorization)

$$(a)\ A=\begin{bmatrix}1 & 2\\ 1 & 0\end{bmatrix} \qquad (b)\ A=\begin{bmatrix}1 & -3 & 0\\ 0 & 1 & 3\\ 2 & -10 & 2\end{bmatrix}$$

Sol: (*a*)

$$A=\begin{bmatrix}1 & 2\\ 1 & 0\end{bmatrix}\xrightarrow{r_{12}^{(-1)}}\begin{bmatrix}1 & 2\\ 0 & -2\end{bmatrix}=U$$

$$\Rightarrow R_{12}^{(-1)}A=U$$

$$\Rightarrow A=(R_{12}^{(-1)})^{-1}U=LU$$

$$\Rightarrow L=(R_{12}^{(-1)})^{-1}=R_{12}^{(1)}=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$$

(b)

$$A=\begin{bmatrix}1 & -3 & 0\\ 0 & 1 & 3\\ 2 & -10 & 2\end{bmatrix}\xrightarrow{r_{13}^{(-2)}}\begin{bmatrix}1 & -3 & 0\\ 0 & 1 & 3\\ 0 & -4 & 2\end{bmatrix}\xrightarrow{r_{23}^{(4)}}\begin{bmatrix}1 & -3 & 0\\ 0 & 1 & 3\\ 0 & 0 & 14\end{bmatrix}=U$$

$$\Rightarrow R_{23}^{(4)}R_{13}^{(-2)}A=U$$

$$\Rightarrow A=(R_{13}^{(-2)})^{-1}(R_{23}^{(4)})^{-1}U=LU$$

$$\Rightarrow L=(R_{13}^{(-2)})^{-1}(R_{23}^{(4)})^{-1}=R_{13}^{(2)}R_{23}^{(-4)}$$

$$=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 2 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & -4 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 2 & -4 & 1\end{bmatrix}$$

- Solving $Ax=b$ with an LU-factorization of A

$$Ax = b \qquad \text{If } A = LU\text{, then } LUx = b$$

$$\text{Let } y = Ux\text{, then } Ly = b$$

- Two steps:

(1) Write $y = Ux$ and solve $Ly = b$ for y

(2) Solve $Ux = y$ for x

- Ex 7: (Solving a linear system using *LU*-factorization)

$$\begin{aligned} x_1 \; - \; 3x_2 \qquad\qquad &= -5 \\ x_2 + 3x_3 &= -1 \\ 2x_1 - 10x_2 + 2x_3 &= -20 \end{aligned}$$

Sol:

$$A = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 2 & -10 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 14 \end{bmatrix} = LU$$

(1) Let $y = Ux$, and solve $Ly = b$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \\ -20 \end{bmatrix} \Rightarrow \begin{aligned} y_1 &= -5 \\ y_2 &= -1 \\ y_3 &= -20 - 2y_1 + 4y_2 \\ &= -20 - 2(-5) + 4(-1) = -14 \end{aligned}$$

(2) Solve the following system $Ux = y$

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \\ -14 \end{bmatrix}$$

So $x_3 = -1$

$$x_2 = -1 - 3x_3 = -1 - (3)(-1) = 2$$

$$x_1 = -5 + 3x_2 = -5 + 3(2) = 1$$

Thus, the solution is

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Keywords in Section 2.4:

- row elementary matrix:
- row equivalent:
- lower triangular matrix:
- upper triangular matrix:
- *LU*-factorization: