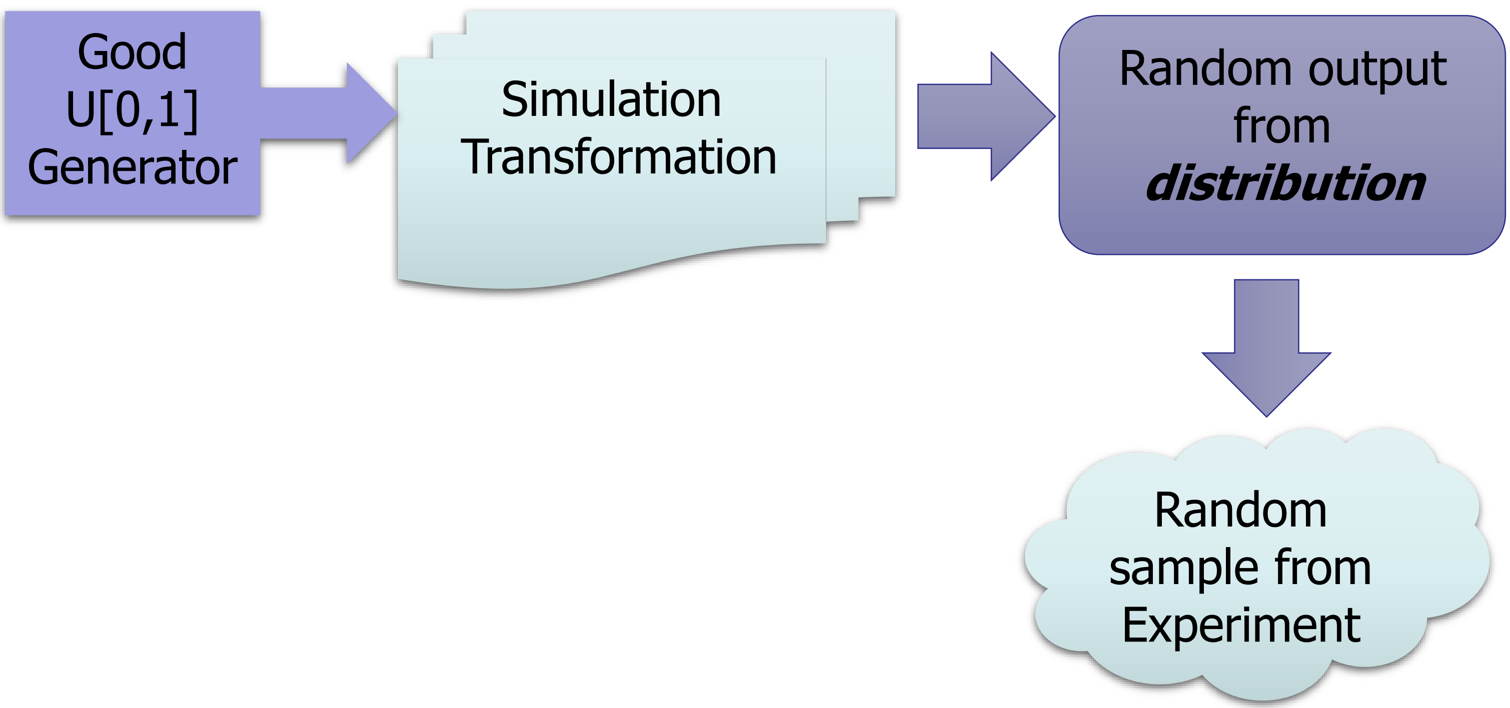
Chapter 5:
Generating Random Numbers from Distributions

Refer to Reading Materials:

Generating Random Numbers

- In simulation, pseudorandom numbers serve as the foundation for generating samples from probability distribution



Generating Random Numbers

- Assume random number generator has been tested and that it produces sequences of $U_i \sim U(0, 1)$
- Take the $U_i \sim U(0, 1)$ to generate from probability distributions.
- We need in simulations many different probability distributions as inputs.
- We need methods for generate random samples from probability distributions using $U_i \sim U(0, 1)$
- The goal is to produce samples X_i from a distribution $F(x)$, given a source of random numbers, $U_i \sim U(0, 1)$.

Generating Random Numbers

- There are four basic strategies or methods for producing random variates:

  1. Inverse transform or inverse CDF method
  2. Convolution
  3. Acceptance/Rejection
  4. Mixture and truncated distributions

1. Inverse Transform

- The method depend on the CDF of the variable.
- Easy method for generating random variates
- The CDF can be easily derived or computed numerically.
- NOT all distribution can be generated using inverse transform method.
- BIG advantage for the inverse transform method: for every U_i, there is ***exactly one*** corresponding X_i from the CDF.

1. Inverse Transform

- First, generate a number $\mathbf{u}_i$ between 0 and 1 (one U-axis) and then find the corresponding $\mathbf{x}_i$ coordinate by using $\mathbf{F}^{-1}(\cdot)$.
- For various values of $\mathbf{u}_i$, the $\mathbf{x}_i$ will be properly "distributed" along the x-axis.
- This method is that there is a one-to-one mapping between $\mathbf{u}_i$ and $\mathbf{x}_i$ because of the monotone property of the CDF.

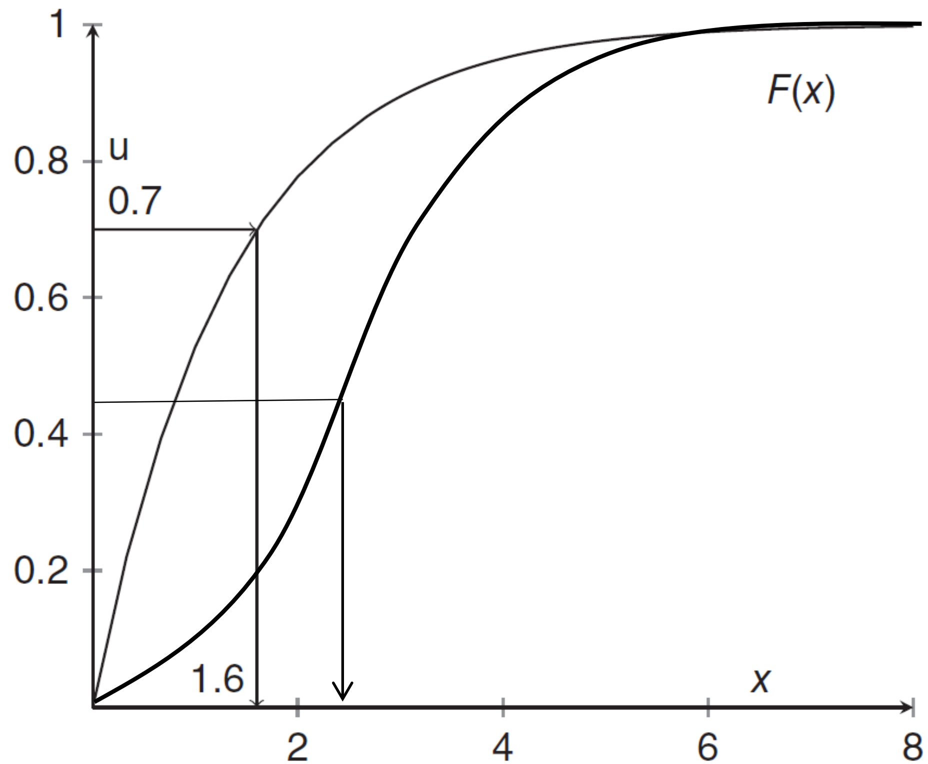
1. Inverse Transform (Continuous Dist.)

Theorem: Let $X \sim F(x)$ **and define** $Y = F(X)$ **,**
then Y is uniformly distributed on (0,1), i.e

$$P\{Y \le y\} = y \quad for\ 0 < y < 1$$

Proof:

$$
\begin{aligned}
P\{Y \le y\} &= P\{F(X) \le y\} \\
&= P\{F^{-1}(F(X)) \le F^{-1}(y)\} \\
&= P\{X \le F^{-1}(y)\} \\
&= F(F^{-1}(y)) \\
&= y
\end{aligned}
$$



1. Inverse Transform (Continuous Dist.)

Exercise:

- Consider the following probability density function:

$$f(x) = \begin{cases} \dfrac{3x^2}{2} & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

a) Derive an inverse transform algorithm for this distribution.

b) Using the following uniform numbers, generate random numbers using from the above distribution

$U_i =$ **0.2379 0.7551 0.2989 0.247 0.3237**

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The CDF :

For $x < -1$, $F(x) = 0$

For $-1 \leq x \leq 1$, $F(x) = \frac{1}{2}(x^3 + 1)$

For $x > 1$, $F(x) = 1$

1. Inverse Transform (Continuous Dist.)

Exercise:

Let $u \sim U[0,1]$ then $u = \mathrm{F}(x)$

$$u = F(x) = \frac{1}{2}(x^3 + 1)$$

$$2u = (x^3 + 1)$$

$$2u - 1 = x^3$$

$$\sqrt[3]{2u - 1} = x$$

Then , $$F^{-1}(u) = \sqrt[3]{2u - 1}$$

1. Inverse Transform (Continuous Dist.)

<u>Exercise:</u>

- U = 0.2379 → $x = \sqrt[3]{2(0.2379) - 1} = -0.8063$
- U = 0.7551 → $x = \sqrt[3]{2(0.7551) - 1} = 0.7991$
- U = 0.2989 → $x = \sqrt[3]{2(0.2989) - 1} = -0.7832$
- U = 0.247 → $x = \sqrt[3]{2(0.2470) - 1} = -0.7969$
- U = 0.3237 → $x = \sqrt[3]{2(0.3237) - 1} = -0.7065$

1. Inverse Transform (Continuous Dist.)

Exponential Distribution

The exponential distribution is often used to model the time until an event until failure and time until an arrival) and has the form:

$$f(x) = \begin{cases} 0.0 & \text{if } x < 0 \\ \lambda e^{-\lambda x} & \text{if } x \geq 0 \end{cases}$$

with

$$\mathrm{E}[X] = \frac{1}{\lambda} \qquad \mathrm{Var}[X] = \frac{1}{\lambda^2}$$

1. Inverse Transform (Continuous Dist.)

Exponential Distribution

- To apply the inverse CDF method, you must first compute the CDF

$$F(x) = P\{X \le x\} = \int_{-\infty}^{x} f(u)du$$

$$= \int_{-\infty}^{0} f(u)du + \int_{0}^{x} f(u)du$$

$$= \int_{0}^{x} \lambda e^{\lambda u} du = -\int_{0}^{x} e^{-\lambda u}(-\lambda)du$$

$$= -e^{\lambda u}\Big|_{0}^{x} = -e^{\lambda x} - (-e^{0}) = 1 - e^{-\lambda x}$$

1. Inverse Transform (Continuous Dist.)

Exponential Distribution

- Now the inverse of the CDF can be derived by setting $u = F(x)$

$$u = 1 - e^{-\lambda x}$$

$$x = \frac{-1}{\lambda} \ln\,(1-u) = F^{-1}(u)$$

- Suppose that $\lambda = 0.75$ and we have $u = 0.7$, then the generated x would be

$$x = (-1/0.75)\ln\,(1 - 0.7) = 1.6053$$

1. Inverse Transform (Continuous Dist.)

Uniform (*a*,*b*)

- The uniform distribution over an interval (a, b)
- Used when the ***analyst does not have much information***
- Assume outcomes are equally likely over a range of values.

$$X \sim \text{Uniform}(a, b)$$

$$f(x) = \begin{cases} \dfrac{1}{b-a} & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = \frac{a+b}{2} \qquad \text{Var}[X] = \frac{(b-a)^2}{12}$$

1. Inverse Transform (Continuous Dist.)

Uniform (*a*,*b*)

- Derive the inverse CDF for the $U(a, b)$ distribution.
- Give a pseudorandom number when $a = 5$, $b = 35$, and $u = 0.25$

$$F(x) = \begin{cases} 0.0 & x < a \\ \dfrac{x-a}{b-a} & a \leq x \leq b \\ 1.0 & x > b \end{cases}$$

1. Inverse Transform (Continuous Dist.)

Uniform (*a*,*b*)

- Derive the inverse CDF for the $U(a, b)$ distribution.

$$u = \frac{x-a}{b-a}$$

$$u(b-a) = x - a$$

$$x = a + u(b-a) = F^{-1}(u)$$

1. Inverse Transform (Continuous Dist.)

Uniform (*a*,*b*)

- Suppose that $a = 5$ and $b = 35$. Suppose that $u = 0.25$, then the generated x would be

$$F^{-1}(u) = x = 5 + 0.25 \times (35 - 5) = 5 + 7.5 = 12.5$$

- ***Algorithm***

1: $u = U(0, 1)$
2: $x = a + u(b - a)$
3: RETURN x

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1. Inverse Transform (Continuous Dist.)

Weibull Distribution

- probability density function:

$$f(x) = \frac{\alpha}{\beta^{\alpha}} x^{\alpha-1} e^{-(x/\beta)^{\alpha}}$$

$$E[X] = \frac{\beta}{\alpha}\Gamma\left(\frac{1}{\alpha}\right) \qquad Var(X) = \frac{\beta^2}{\alpha^2}\left\{2\Gamma\left(\frac{2}{\alpha}\right) - \frac{1}{\alpha}\left[\Gamma\left(\frac{1}{\alpha}\right)\right]^2\right\}$$

- CDF

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - e^{(-x/\beta)^{\alpha}} & \text{if } x \geq 0 \end{cases}$$

1. Inverse Transform (Continuous Dist.)

Weibull Distribution

- CDF

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - e^{(-x/\beta)^{\alpha}} & \text{if } x \geq 0 \end{cases}$$

$$U = 1 - e^{(-x/\beta)^{\alpha}} \qquad \text{if } x \geq 0$$

$$1 - U = e^{(-x/\beta)^{\alpha}}$$

$$\ln(1 - U) = \left(-\frac{x}{\beta}\right)^{\alpha}$$

$$[\ln(1 - U)]^{\frac{1}{\alpha}} = -\frac{x}{\beta}$$

$$-\beta[\ln(1 - U)]^{\frac{1}{\alpha}} = x$$

1. Inverse Transform (Discrete Dist.)

Example

Cars arrive to a gas station at random and they spend a random amount of fuel based on the size and the type of car. The work hours of the station is from 8 am to 6 pm.

- It is assumed that the time between car arrival is Exponentially distributed with mean 15 min
- The amount of fuel is uniform distribution between 10 liters to 70 liter.

1. The gas station wants simulate the behavior of the station during 3 hours.
2. From (1), what is the average of number of arrivals per hour.

1. Inverse Transform (Discrete Dist.)

Example (Cont.)

3. From (1), what is the average of amount of money collected per hour, given that the gas station charge 0.85 SR per Liter
4. What is the probability of fueling more than 15 liters in 3 hours.
5. Answer 1,2,3,4 for one full day?
6. From 5, what is the distribution of number of arrivals in one hour?