4.4 Cauchy- Euler Differential equation

A Cauchy Euler differential equation is the form

$$a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \ldots + a_1 x y' + a_0 y = 0.$$

Homog

$x=e^t$ $t=\ln x$ $x>0$ $y=x^m$ $x=-\xi$ $\xi<0$

Exercises (4.4)

In problems 1 through 20 find the general solution of the following differential equations , where we suppose that $x > 0$.

1) $x^2y'' - y = 0.$
2) $xy'' + y' = 0.$
3) $xy'' - y' = 0.$
4) $4x^2y'' + y = 0.$
5) $x^2y'' + 5xy' + 3y = 0.$
6) $x^2y'' + xy' + 4y = 0.$
7) $x^2y'' - 3xy' - 2y = 0.$
8) $25x^2y'' + 78xy' + y = 0.$
9) $4x^2y'' + 4xy' - y = 0.$
10) $x^2y'' + 5xy' + 4y = 0.$
11) $x^2y'' - xy' + 2y = 0.$
12) $x^2y'' + 9xy' + 6y = 0.$
13) $x^2y'' - 7xy' + 41y = 0.$
14) $3x^2y'' + 6xy' + y = 0.$
15) $2x^2y'' + xy' + y = 0.$
16) $x^2y'' - 6y = 0.$
17) $x^2y'' + xy' - y = 0.$
18) $x^2y'' - 2x^2y' - 2xy' + 8y = 0.$
19) $x^2y''' - 2x^2y' + 4xy' - 6y = 0.$
20) $x^3y''' + 4x^2y'' - 8xy' + 8y = 0.$

21) In problems 1 through 12, solve the given differential equations by the substitution $x = e^t$.

28) $\begin{cases} x^2y'' - xy' + y = 0 \\ y(-1) = 1 \ , \ y'(-1) = 0 \end{cases}$ $x<0$ $X=-x$ $X>0$

In Exercises 41 through 45, find the general solutions of the differential equations .

41) $(x-1)^2y'' + 5(x-1)y' + 4y = 0 \quad ; \quad x > 1.$
42) $(x+3)^2y'' + 3(x+3)y' + 5y = 0 \quad ; \quad x < -3.$
43) $(x-2)^2y'' - (x-2)y' + y = 0 \quad ; \quad x > 2.$
44) $(3x+4)^2y'' + 10(3x+4)y' + 9y = 0 \quad ; \quad 3x+4 > 0.$
45) $(x+2)^2y'' + (x+2)y' + y = 0 \quad ; \quad x < -2 .$

5) $y = x^m \to y' = mx^{m-1}$, $y'' = m(m-1)x^{m-2}$

in D.E: $m(m-1)x^m + 5mx^m + 3x^m = 0$

$[m^2 - m + 5m + 3]x^m = 0$

$m^2 + 4m + 3 = 0$ $\quad \div x^m (\neq 0)$

$(m+3)(m+1) = 0$

$m = -3, m = -1$

$y = y_c = c_1 e^{-3x} + c_2 e^{-x}$ ×

$y = y_c = c_1 x^{-3} + c_2 x^{-1}$ ✓

6) $y = x^m$, $y' = mx^{m-1}$, $y'' = m(m-1)x^{m-2}$

in D.E: $m(m-1)x^m + mx^m + 4x^m = 0$

$[m^2 - m + m + 4]x^m = 0$

$m^2 + 4 = 0$ $\quad \div x^m (\neq 0)$

$m = 2i, m = -2i$

$y = y_c = c_1 \cos(2x) + c_2 \sin(2x)$ ×

$y = y_c = c_1 \cos(2\ln x) + c_2 \sin(2\ln x)$ ✓

10) $y = x^m$, $y' = mx^{m-1}$, $y'' = m(m-1)x^{m-2}$

in D.E: $m(m-1)x^m + 5mx^m + 4x^m = 0$

$[m^2 - m + 5m + 4]x^m = 0$

$m^2 + 4m + 4 = 0$ $\quad (\div x^m \neq 0)$

$(m+2)(m+2) = 0$

$m = -2$

$y = y_c = c_1 x^{-2} + c_2 \ln x \cdot x^{-2}$

11) $x^2y'' - xy' + 2y = 0$

$y = x^m$, $y' = mx^{m-1}$, $y'' = m(m-1)x^{m-2}$

in D.E: $m(m-1)x^m - mx^m + 2x^m = 0$

$[m^2 - m - m + 2]x^m = 0$

$m^2 - 2m + 2 = 0$ $\quad (\div x^m \neq 0)$

$m = \frac{2 \pm \sqrt{4 - 4 \cdot 1 \cdot 2}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$

$y = y_c = x^1\left[c_1 \cos(1 \cdot \ln x) + c_2 \sin(1 \cdot \ln x)\right]$

17) $x^3y''' + xy' - y = 0$

$y = x^m$, $y' = mx^{m-1}$, $y'' = m(m-1)x^{m-2}$

$y''' = m(m-1)(m-2)x^{m-3}$

in D.E: $m(m-1)(m-2)x^m + mx^m - x^m = 0$

$[m(m^2 - 2m + 2) + m - 1]x^m = 0$

$[m^3 - 3m^2 + 3m - 1] = 0$ $\quad (\neq x^m \div)$

$m^3 - 1 + 3m(1-m) = 0$

$(m-1)(m^2 + m + 1) - 3m(m-1) = 0$

$(m-1)[m^2 + m + 1 - 3m] = 0$

$(m-1)(m^2 - 2m + 1) = 0$

$(m-1)(m-1)(m-1) = 0$

$m = 1$

$y = y_c = c_1 x^1 + c_2 \ln x \cdot x^1 + c_3 (\ln x)^2 \cdot x^1$

(21) #12: $x^3y''' + 3x^2y'' + 6xy' + 6y = 0$

$y' = \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt} \cdot \frac{1}{x}$ $\quad x = e^t \to \ln x = t$

$y'' = \frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} \cdot \frac{-1}{x^2} + \frac{d}{dx}\left(\frac{dy}{dt}\right) \cdot \frac{1}{x}$

$= \frac{-1}{x^2}\frac{dy}{dt} + \frac{d^2y}{dt^2}\frac{dt}{dx} \cdot \frac{1}{x}$

$= -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x^2}\frac{d^2y}{dt^2} = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$

in D.E: $\frac{d^3y}{dt^3} - \frac{d^2y}{dt^2} + 3\frac{d^2y}{dt^2} + 6y = 0$

$\frac{d^3y}{dt^3} + 2\frac{d^2y}{dt^2} + 6y = 0$

(c.e): $m^3 + 2m + 6 = 0$

$(m+6)(m+1) = 0$

$m = -6, m = -1$

$y = y_c = c_1 e^{-6t} + c_2 e^{-t}$

$= c_1 e^{-6\ln x} + c_2 e^{-\ln x}$

$= c_1 \cdot x^{-6} + c_2 x^{-1}$

41) $(x-1)^2y'' + 5(x-1)y' + 4y = 0, \ x > 1$

$y = (x-1)^m$, $y' = m(x-1)^{m-1}$, $y'' = m(m-1)(x-1)^{m-2}$

in D.E: $m(m-1)(x-1)^m + 5m(x-1)^m + 4(x-1)^m = 0$

$[m^2 - m + 5m + 4](x-1)^m = 0$

$m^2 + 4m + 4 = 0$ $\quad \div (x-1)^m \neq 0$

$(m+2)(m+2) = 0$

$m = -2$

$y = y_c = c_1 x^{-2} + c_2 \ln x \cdot x^{-2}$ ×

$y = y_c = c_1 (x-1)^{-2} + c_2 \ln(x-1)(x-1)^{-2}$ ✓

45) $y = c_1 \cos\ln[-(x+2)] + c_2 \sin\ln[-(x+2)]$