Let $0 < a_1 < b_1$ and define

$a_{n+1} = \sqrt{a_n b_n}$, $b_{n+1} = \frac{a_n + b_n}{2}$.

1) prove that the sequences $\{a_n\}$ and $\{b_n\}$ each converge.
2) prove that they have the same limit.

Problem 4.2:

$$\sqrt{ab} \le \frac{a+b}{2}$$

$$\boxed{\sqrt{a_n b_n} \le \frac{a_n + b_n}{2}}$$

$$\boxed{a_{n+1} \le b_{n+1}} \Rightarrow 1 \le \frac{b_{n+1}}{a_{n+1}}$$

$$a_2 = \sqrt{a_1 b_1} \ge \sqrt{a_1 \cdot a_1} = a_1 \quad (a_n \uparrow ?)$$

$$b_2 = \frac{a_1 + b_1}{2} \le \frac{b_1 + b_1}{2} = b_1 \quad (b_n \downarrow ?)$$

$$\frac{a_{n+1}}{a_n} = \frac{\sqrt{a_n b_n}}{a_n} = \sqrt{\frac{b_n}{a_n}} \ge \sqrt{1} = 1$$

$\therefore (a_n)$ is increasing.

$$\frac{b_{n+1}}{b_n} = \frac{\frac{a_n + b_n}{2}}{b_n}$$

$$= \frac{1}{2}\left[\frac{a_n}{b_n} + 1\right] \le \frac{1}{2}\left(1+1\right] = 1$$

$\therefore (b_n)$ is decreasing.

(a_n) increasing $\quad$ (b_n) decreasing

$$a_1 \le a_n \le a_{n+1} \le b_{n+1} < b_n \le b_1$$

(a_n) is bounded above by b_1

(b_n) is bounded below by a_1

$\therefore (a_n)$ is convergent.

and (b_n) is convergent.

(2): Assume that $\lim a_n = l$

$\Rightarrow \lim\limits_{n\to\infty} b_n = m \quad$ (Want $l = m$??)

$\Rightarrow \lim\limits_{n\to\infty} a_{n+1} = l$ and $\lim\limits_{n\to\infty} b_{n+1} = m$

$\Rightarrow \lim \sqrt{a_n b_n} = l$ & $\lim \frac{a_n + b_n}{2} = m$

$\Rightarrow \lim \sqrt{a_n} \cdot \lim \sqrt{b_n} = l$

& $\frac{\lim a_n + \lim b_n}{2} = m$

$\Rightarrow \sqrt{\lim a_n} \cdot \sqrt{\lim b_n} = l$

& $\frac{l + m}{2} = m \quad \ldots\ldots (1)$

$\sqrt{l} \cdot \sqrt{m} = l \quad \ldots\ldots (2)$

$(1) \Rightarrow \frac{l}{2} + \frac{m}{2} = m \Rightarrow \frac{l}{2} = \frac{m}{2}$

$\Rightarrow \boxed{l = m}$

$(2) \Rightarrow l \cdot m = l^2$

$l \cdot m - l^2 = 0$

$l(m - l) = 0$

$\Rightarrow l = 0$ or $m - l = 0$

$m = l$