4.2 :

14) $xy'' + y' = 0$; $y_1 = \ln x$ and $x > 0$

Use formula to find y_2

Solution: $y_2 = y_1 \cdot \int \frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx$ (formula)

$= \ln x \cdot \int \frac{e^{-\int \frac{1}{x}dx}}{(\ln x)^2}\,dx$

$= \ln x \cdot \int \frac{e^{-\ln x}}{(\ln x)^2}\,dx$

$= \ln x \cdot \int \frac{1}{x(\ln x)^2}\,dx$ $\qquad u = \ln x$, $du = \frac{1}{x}dx$

$= \ln x \cdot \frac{-1}{\ln x}$ $\qquad \int \frac{du}{u^2} = \int u^{-2}\,du = \frac{u^{-1}}{-1} = \frac{-1}{\ln x}$

$= -1$

G.S : $y = c_1 y_1 + c_2 y_2$

$= c_1 \cdot \ln x + c_2(-1)$

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(26) $5x^2y'' - 3xy' + 3y = x^{\frac{1}{2}}$, $y_1 = e^x$

Put $y = y_1 \cdot u$ $\quad (u = ?)$

general solution $y = e^x \cdot u$

$y' = e^x u + e^x u'$

$y'' = e^x u + e^x u' + e^x u'' + e^x u'$

$= e^x u'' + 2e^x u' + e^x u$

Go to D.E : $5x^2 e^x u'' + 10x^2 e^x u' + 5x^2 e^x u$

$-3x e^x u - 3x e^x u' + 3e^x u = x^{\frac{1}{2}}$

$5x^2 \cancel{e^x} u'' + (10x^2 - 3x)\cancel{e^x} u' + (5x^2 - 3x + 3)\cancel{e^x} u = x^{\frac{1}{2}} e^{-x}$

Put $u' = v \rightarrow u'' = v'$

$5x^2 v' + (10x^2 - 3x)v + (5x^2 - 3x + 3)$

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(23) $y'' + y = \sec x$, $y_1 = \sin x$

Put $y = y_1 \cdot u$

$y = \sin x \cdot u$

$y' = \cos x \cdot u + \sin x \cdot u'$

$y'' = -\sin x\, u + \cos x \cdot u' + \cos x \cdot u' + \sin x \cdot u''$

$= \sin x\, u'' + 2\cos x\, u' - \sin x \cdot u$

Go to D.E : $\sin x\, u'' + 2\cos x\, u' - \cancel{\sin x\, u} + \cancel{\sin x \cdot u} = \sec x$

Put $u' = v \rightarrow u'' = v'$

$\sin x \cdot v' + \cancel{2\cos x}\, v = \cancel{\frac{\sec x}{\sin x}}$

$P(x) = \frac{2\cos x}{\sin x}$

$\mu = e^{\int P(x)dx} = e^{\int \frac{2\cos x}{\sin x}dx} = e^{2\ln \sin x} = \sin^2 x$

$v = \frac{1}{\mu}\left[\int \mu \cdot \frac{\sec x}{\sin x}\,dx + c\right]$

$v = \frac{1}{\sin^2 x}\left[\int \sin^2 x \cdot \frac{\sec x}{\cancel{\sin x}}\,dx + c\right]$

$v = \frac{1}{\sin^2 x}\left[\int \frac{\sin x}{\cos x}\,dx + c\right]$

$v = \frac{1}{\sin^2 x}\left[-\ln(\cos x) + c\right]$

$v = \frac{-\ln(\cos x)}{\sin^2 x} + c\csc^2(x)$

$u' = v = \frac{-\ln(\cos x)}{\sin^2 x} + c\csc^2(x)$

$u = \int \left[\frac{-\ln(\cos x)}{\sin^2 x} + c\csc^2(x)\right]dx$ $\quad u$: LIATE

$\frac{-\ln(\cos x)}{\sin^2 x} \rightarrow$ by parts

$uv - \int v\,du$ $\quad \Big| \quad u = -\ln(\cos x)$

$+\ln(\cos x)\cot(x)$ $\quad \Big| \quad du = \frac{\sin x}{\cos x}dx$

$+\int \cot(x)\frac{\sin x}{\cos x}dx$ $\quad \Big| \quad dv = \frac{1}{\sin^2(x)}dx$

$= \cot(x)\ln(\cos x) + x - c\cot(x)$

$u = \cot(x)\ln(\cos x) + x - c \cdot \cot(x) + c_2$

G.S is $y = \cos(x)\ln(\cos x) + x\sin(x) - c\cos(x) + c_2\sin(x)$