Available online at www.sciencedirect.com
ScienceDirect
Computational Statistics & Data Analysis 51 (2007) 5377 – 5387
COMPUTATIONAL STATISTICS & DATA ANALYSIS
www.elsevier.com/locate/csda

Mixture of two inverse Weibull distributions: Properties and estimation

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Available online 9 October 2006

Abstract

The mixture model of two Inverse Weibull distributions (MTIWD) is investigated. First, some properties of the model with some graphs of the density and hazard function are discussed. Next, the identifiability property of the MTIWD is proved. In addition, the estimates of the unknown parameters via the EM Algorithm are obtained. The performance of the findings in the paper is showed by demonstrating some numerical illustrations through Monte Carlo simulations.

Keywords: Finite mixtures; Mixing proportion; Mode; Median; Reliability and failure rate functions; Identifiability; EM Algorithm; Mean square error; Bias; Monte Carlo method

1. Introduction

Mixture models play a vital role in many practical applications. For example, direct applications of finite mixture models are in fisheries research, economics, medicine, psychology, palaeoanthropology, botany, agriculture, zoology, life testing and reliability, among others. Indirect applications include outliers, Gaussian sums, cluster analysis, latent structure models, modeling prior densities, empirical Bayes method and nonparametric (kernel) density estimation. In many applications, the available data can be considered as data coming from a mixture population of two or more distributions. This idea enables us to mix statistical distributions to get a new distribution carrying the properties of its components.

The mixture of two Inverse Weibull distribution (MTIWD) has its pdf as

$$f(t;\Theta) = p_1 f_1(t;\Theta_1) + p_2 f_2(t;\Theta_2), \quad p_1 + p_2 = 1, \tag{1.1}$$

where $\Theta = (p_1, \alpha_1, \alpha_2, \beta_1, \beta_2)$, $\Theta_i = (\alpha_i, \beta_i)$, $i = 1, 2$, and $f_i(t;\Theta_i)$, the density function of the ith component, is given by

$$f_i(t;\Theta_i) = \beta_i \alpha_i^{-\beta_i} t^{-(\beta_i+1)} \mathrm{e}^{-(\alpha_i t)^{-\beta_i}}, \quad t \geqslant 0, \quad \alpha_i, \beta_i > 0, \quad i = 1, 2. \tag{1.2}$$

The cdf of the MTIWD is given by

$$F(t;\Theta) = p_1 F_1(t;\Theta_1) + p_2 F_2(t;\Theta_2), \tag{1.3}$$

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doi:10.1016/j.csda.2006.09.016

where $F_i(t;\Theta_i)$, the cdf of the ith component, is given by

$$F_i(t;\Theta_i)=\mathrm{e}^{-(\alpha_i t)^{-\beta_i}},\quad t\geqslant 0,\quad \alpha_i,\beta_i>0,\ i=1,2. \tag{1.4}$$

Mixture distributions have been considered extensively by many authors; for an excellent survey of estimation techniques, discussion and applications, see Everitt and Hand (1981), Titterington et al. (1985), Maclachlan and Basford (1988), Lindsay (1995), Maclachlan and Krishnan (1997), and Maclachlan and Peel (2000). Recently, AL-Hussaini and Sultan (2001) have reviewed properties and the estimation techniques of finite mixtures of some life time models.

Identifiability questions of mixtures must be settled before one can meaningfully discuss the problems of estimation, testing hypotheses or classification of random variables, which are based on observations from the mixture. Identifiability gives a unique representation for a class of mixtures. Lack of identifiability is a serious problem if we intend to classify future observations into one of several classes from our knowledge of the component distributions. Identifiability of mixtures has been discussed by several authors, including Teicher (1963), Yakowitz and Spragins (1968), Balakrishnan and Mohanty (1972), AL-Hussaini and Ahmad (1981), Ahmad and AL-Hussaini (1982), and Ahmad (1988).

Jiang et al. (1999) have shown that the Inverse Weibull (IW) mixture models with negative weight can represent the output of a system under certain situations. Jiang et al. (2001) have considered the shapes of the density and failure rate functions and graphical methods to discuss the MTIWD. Jiang et al. (2003) have discussed the aging property of the unimodal failure rate models including the IW distribution. Calabria and Pulcini (1990) have discussed the maximum likelihood and least square estimates of the parameters of the IW distribution.

In this paper we discuss some important measures of the MTIWD. Also, we show that the MTIWD is identifiable. In addition, we estimate the vector of the unknown parameters Θ of a mixture model via the EM Algorithm proposed by Dempster et al. (1977). Further we carry out some simulated illustrations using Monte Carlo method.

The remainder of this paper has the following organization. In Section 2, we summarize and discuss some properties of the MTIWD. These results play a significant role in the development of statistical methods based on the pdf of the MTIWD given in (1.1) and (1.2). In Section 3, we use the EM Algorithm to estimate the vector of the five parameters of the pdf of the MTIWD given in (1.1) and (1.2). In Section 4, we carry out some simulation studies to illustrate the estimation technique considered in Section 3. Finally, we draw conclusion in Section 5.

2. Properties

Keller and Kamath (1982) and Jiang et al. (2001) have discussed some properties of the pdf of the IW distribution given in (1.2). In this section, we derive and analyze some properties for the MTIWD by extending the corresponding results of the IW distribution as follows:

1. *Mean and variance*: The mean of the pdf of the MTIWD given in (1.1) and (1.2) is

$$E(T)=\frac{p_1}{\alpha_1}\Gamma\left(1-\frac{1}{\beta_1}\right)+\frac{p_2}{\alpha_2}\Gamma\left(1-\frac{1}{\beta_2}\right),\quad \beta_1,\beta_2>1, \tag{2.1}$$

while the variance is given by

$$\begin{aligned}Var(T)=&\frac{p_1}{\alpha_1^2}\left[\Gamma\left(1-\frac{2}{\beta_1}\right)-p_1\Gamma^2\left(1-\frac{1}{\beta_1}\right)\right]+\frac{p_2}{\alpha_2^2}\left[\Gamma\left(1-\frac{2}{\beta_2}\right)\right.\\&\left.-\,p_2\Gamma^2\left(1-\frac{1}{\beta_2}\right)\right]-\frac{2p_1p_2}{\alpha_1\alpha_2}\left[\Gamma\left(1-\frac{1}{\beta_1}\right)\Gamma\left(1-\frac{1}{\beta_2}\right)\right],\\&\beta_1,\beta_2>2,\end{aligned} \tag{2.2}$$

where $\Gamma(\cdot)$ is the gamma function.

2. *Mode and median*: The mode (modes) of the MTIWD is (are) obtained by solving the following nonlinear equation with respect to t

$$\sum_{i=1}^{2}p_i\beta_i\alpha_i^{-\beta_i}t^{-(\beta_i+2)}\mathrm{e}^{-(\alpha_i t)^{-\beta_i}}[-(\beta_i+1)+\beta_i\alpha_i^{-\beta_i}t^{-(\beta_i)}]=0. \tag{2.3}$$

Table 1
The mode(s) and median of the MTIWD

$\Theta = (p_1, \alpha_1, \alpha_2, \beta_1, \beta_2)$	Mode(s)	Median
0.2,1,2,2,3	0.4592	0.6289
0.4,1,2,2,3	0.4689	0.7342
0.6,1,2,2,3	0.4996	0.8830
0.2,2.5,1,2,2.9	0.3266, 0.8861	1.0398
0.4,2.5,1,2,2.9	0.3266, 0.8575	0.9212
0.6,2.5,1,2,2.9	0.3266, 0.7879	0.7622

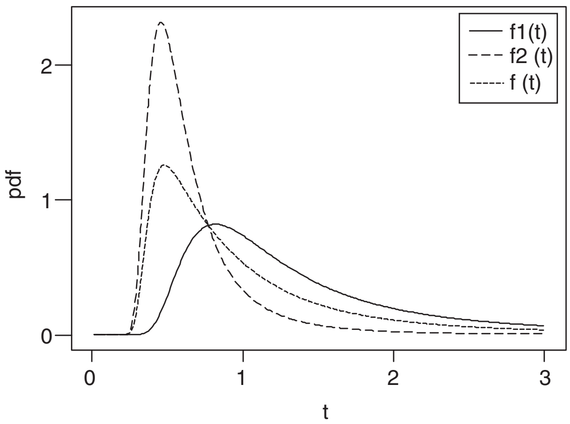

Fig. 1. Density functions: components and their mixture with parameters (0.5, 1.0, 2.0, 2.0, 3.0).

By using (1.3) and (1.4), the median is obtained by solving the following nonlinear equation with respect to t

$$p_1 \mathrm{e}^{-(\alpha_1 t)^{-\beta_1}} + p_2 \mathrm{e}^{-(\alpha_2 t)^{-\beta_2}} = 0.5. \tag{2.4}$$

Table 1 displays the mode and median of the MTIWD based on different choices of the parameters.

The values of the parameters p_1, α_1, α_2, β_1 and β_2, in Table 1 are chosen to demonstrate the unimodal and bimodal cases for the probability density function of the mixture model. From Table 1, we see that the mode is slightly affected by the variation in the values of the mixing proportion p_1, while one mode is stable in the bimodal case. In addition, for the unimodal case, the median increases when p_1 increases. Conversely, for the bimodal case, we note that the median decreases when p_1 increases. Figs. 1 and 2 show two different shapes of the probability density function of the MTIWD.

3. *Reliability and failure rate functions*: The reliability function (survival function) of the MTIWD is given by

$$R(t) = p_1(1 - \mathrm{e}^{-(\alpha_1 t)^{-\beta_1}}) + p_2(1 - \mathrm{e}^{-(\alpha_2 t)^{-\beta_2}}). \tag{2.5}$$

By using (1.3) and (1.4) it can be seen that the failure rate function (hazard rate function, HRF) of the MTIWD is given by

$$r(t) = \frac{p_1\beta_1\alpha_1^{-\beta_1} t^{-(\beta_1+1)} \mathrm{e}^{-(\alpha_1 t)^{-\beta_1}} + p_2\beta_2\alpha_2^{-\beta_2} t^{-(\beta_2+1)} \mathrm{e}^{-(\alpha_2 t)^{-\beta_2}}}{p_1(1 - \mathrm{e}^{-(\alpha_1 t)^{-\beta_1}}) + p_2(1 - \mathrm{e}^{-(\alpha_2 t)^{-\beta_2}})}, \tag{2.6}$$

which can be written in view of the result by AL-Hussaini and Sultan (2001) as

$$r(t) = h(t) r_1(t) + (1 - h(t)) r_2(t). \tag{2.7}$$

The derivative of the HRF is given by

$$r'(t) = h(t) r_1'(t) + (1 - h(t)) r_2'(t) - h(t)(1 - h(t))[r_1(t) - r_2(t)]^2, \tag{2.8}$$

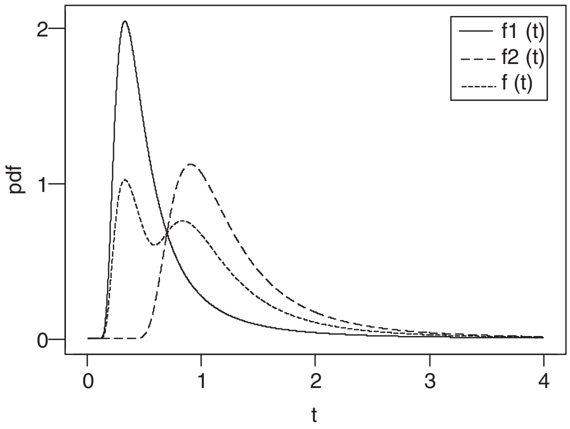

Fig. 2. Density functions: components and their mixture with parameters (0.5, 2.5, 1.0, 2.0, 2.9).

where for $i = 1, 2$

$$h(t) = \frac{1}{1 + \dfrac{p_2 R_2(t)}{p_1 R_1(t)}}, \quad r_i(t) = \frac{f_i(t)}{R_i(t)} \quad \text{and} \quad R_i(t) = 1 - \mathrm{e}^{-(\alpha_i t)^{-\beta_i}}. \tag{2.9}$$

The failure rate function of the MTIWD given in (2.6) satisfies the following limits.

Lemma 1.

$$\lim_{t \to 0} r(t) = 0 \tag{2.10}$$

and

$$\lim_{t \to \infty} r(t) = 0. \tag{2.11}$$

Proof. By using the Taylor expansion, we can express $r_i(t)$ given in (2.9) as

$$r_i(t) = \frac{\beta_i}{\left[t + \dfrac{1}{2} \dfrac{1}{\alpha_i^{\beta_i} t^{\beta_i - 1}} + \dfrac{1}{6} \dfrac{1}{\alpha_i^{2\beta_1} t^{2\beta_i - 1}} + \cdots \right]}, \quad i = 1, 2. \tag{2.12}$$

The denominator in (2.12) tends to infinity as $t \to 0$, and so

$$\lim_{t \to 0} r_1(t) = 0 \quad \text{and} \quad \lim_{t \to 0} r_2(t) = 0. \tag{2.13}$$

From (2.9), it can be shown that

$$\lim_{t \to 0} h(t) = p_1 \tag{2.14}$$

and hence (2.10) is proved. □

Once again, from (2.9), we note that $\frac{p_2 R_2(t)}{p_1 R_1(t)} \geqslant 0$, hence $\lim_{t \to \infty} \frac{p_2 R_2(t)}{p_1 R_1(t)} \neq -1$. It follows that $|h(t)| < \infty$. Moreover from (2.12), it can be shown that

$$\lim_{t \to \infty} r_i(t) = 0 \quad \text{for } i = 1, 2, \tag{2.15}$$

and hence (2.11) is proved. □

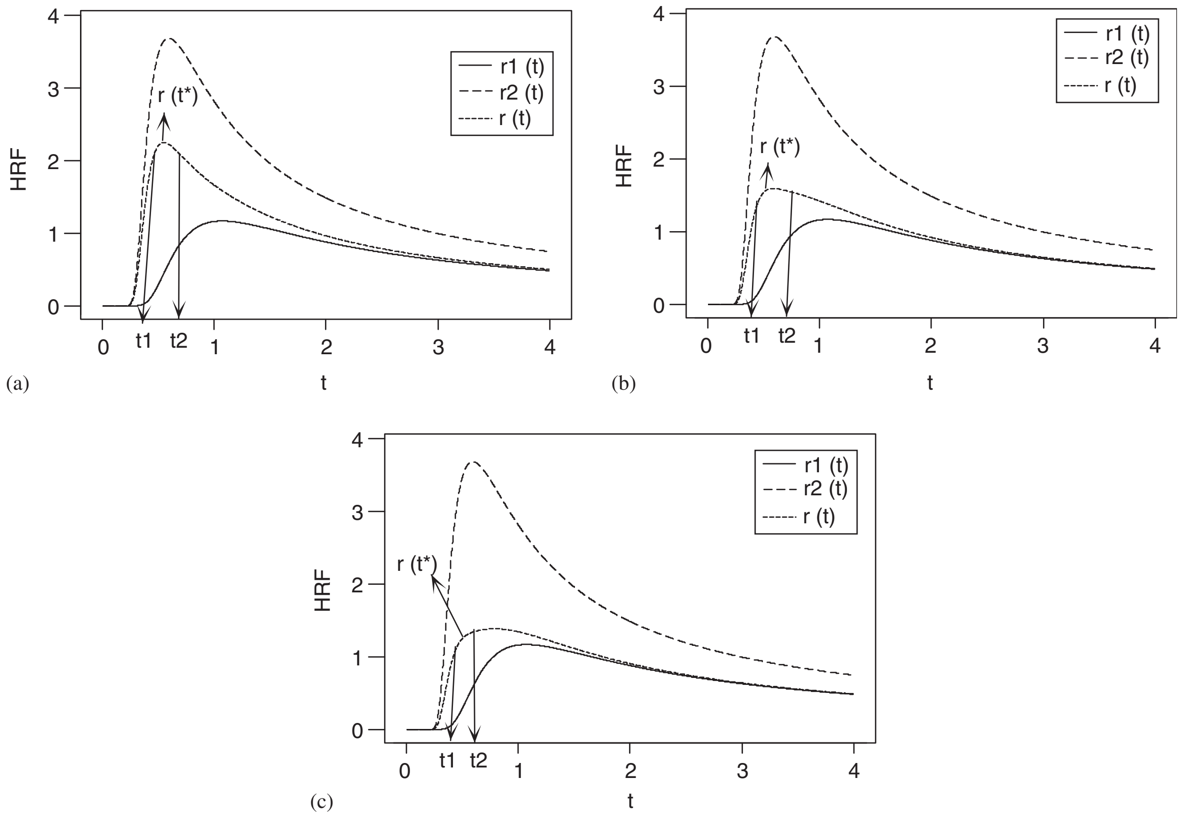

Fig. 3. HR functions components and their mixture with parameters (a) (0.3, 1.0, 2.0, 2.0, 3.0), (b) (0.5, 1.0, 2.0, 2.0, 3.0), (c) (0.6, 1.0, 2.0, 2.0, 3.0).

4. *Interpretation of the failure rate curves*: Suppose that $t_1 = \min(t_1^*, t_2^*)$ and $t_2 = \max(t_1^*, t_2^*)$, where for $i = 1, 2$, t_i^* represents the mode of the density function $f_i(t)$. From $r_i(t) = \frac{f_i(t)}{R_i(t)}$, we see that both of $f_1(t)$ and $f_2(t)$ in the numerator of $r_i(t)$ increase on $(0, t_1)$, whereas the denominator decreases on the same interval. Therefore, $r(t)$ increases on $(0, t_1)$. Also, as $t \to \infty$, $r(t) \to 0$. Within the interval (t_1, ∞), two cases arise:

(a) *Unimodal case*: Suppose that t^* is the maximum point of the failure rate mixture. When the difference Δ between $r_1(t)$ and $r_2(t)$ on the interval (t_1, t^*) is so small that the first two terms of $r'(t)$ in (2.8) dominate the third term, then $r'(t) > 0$ on (t_1, t^*). Then, the difference Δ increases to the point that the third term in $r'(t)$ dominates the first two terms and $r'(t) < 0$ on (t^*, ∞). Summarizing, the failure rate of the MTIWD increases on $(0, t^*)$ and decreases on (t^*, ∞), reaching zero as $t \to \infty$. See Figs. 3(a–c).

(b) *Bimodal case*: Suppose that t^* and t^{**} denote, respectively, the smallest and largest maximum point of the failure rate mixture. When the difference Δ between $r_1(t)$ and $r_2(t)$, on the interval (t_1, t^*), is small where $t_1 < t^* < t_2 < t^{**}$, then the third term of (2.8) is dominated by the first two terms and hence $r'(t) > 0$ on $(0, t^*)$. The difference Δ on the interval (t^*, t^{***}), where t^{***} is the local minimum point of $r(t)$ becomes larger to the point that the third term in $r'(t)$ dominates the first two terms and hence, $r'(t) < 0$ on (t^*, t^{***}). On (t^{***}, t^{**}), the difference becomes small so that the third term in $r'(t)$ is dominated by the first two terms, therefore, $r'(t) > 0$. Summarizing, the failure rate of the mixed model increases on $(0, t^*)$, decreases on (t^*, t^{***}), increases on (t^{***}, t^{**}) and decreases again on (t^{**}, ∞), reaching 0 as t tends to ∞, see Figs. 4(a–c).

5. *Identifiability*: Chandra (1977) has proved the following: Let ϕ be a transform associated with each $F_i \in \Phi$ having the domain of definition D_{ϕ_i} with linear map $M : F_i \to \phi_i$. If there exists a total ordering ($\leqslant$) of Φ

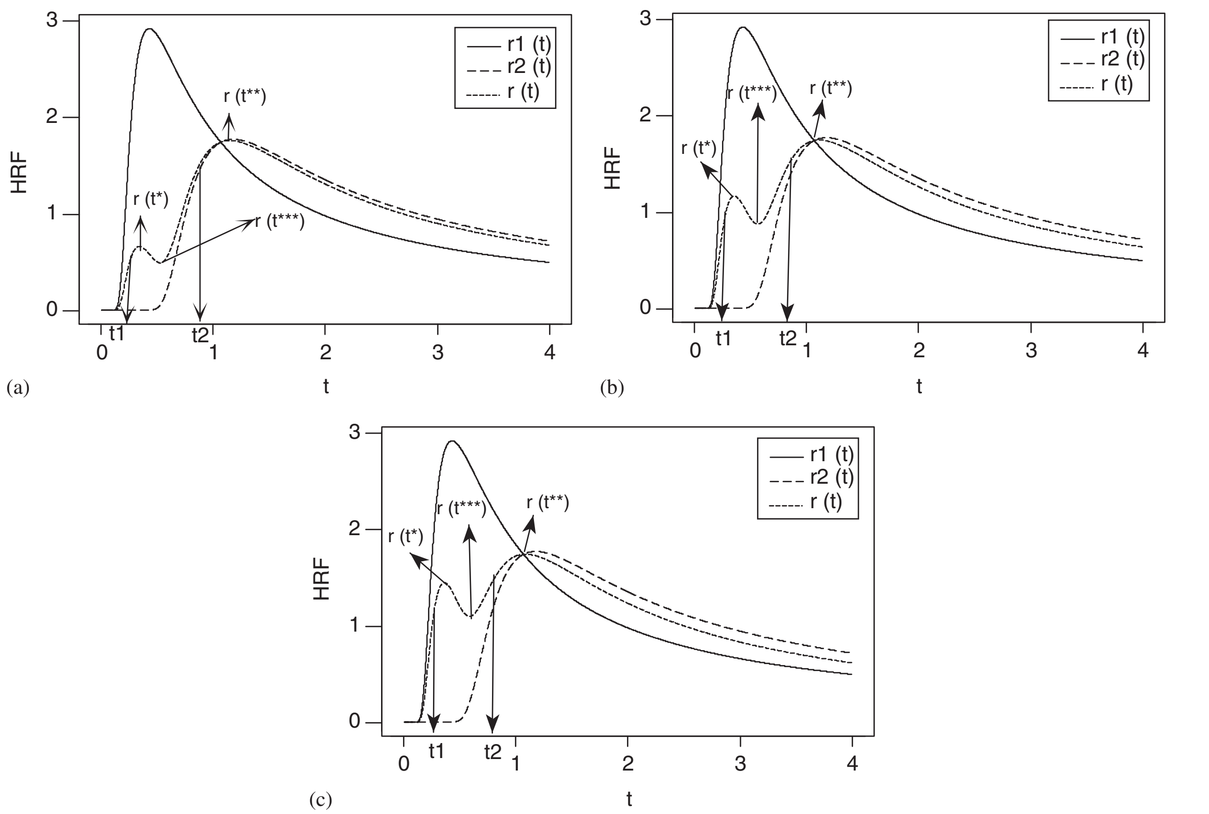

Fig. 4. HR functions components and their mixture with parameters (a) (0.3, 2.5, 1.0, 2.0, 2.9), (b) (0.5, 2.5, 1.0, 2.0, 2.9), (c) (0.6, 2.5, 1.0, 2.0, 2.9).

such that

(i) $F_1 \leqslant F_2$, $(F_1, F_2 \in \Phi)$ implies $D_{\phi_1} \subseteq D_{\phi_2}$;
(ii) for each $F_1 \in \Phi$, there exists some $s_1 \in D_{\phi_1}$, $\phi_1(s) \neq 0$ such that $\lim_{s \to s_1} \phi_2(s)/\phi_1(s) = 0$ for $F_1 < F_2$, $(F_1, F_2 \in \Phi)$;

then the class Λ of all finite mixing distributions is identifiable relative to Φ.

By using Chandra's approach, we prove the following proposition.

Proposition. *The class of all finite mixing distributions relative to the IW distribution is identifiable.*

Proof. Let T be a random variable having the pdf and cdf of the IW distribution given in (1.2) and (1.4), respectively. Then the sth moments of the ith IW component are given by

$$\phi_i(s) = E(T^s) = \alpha_i^{-s} \Gamma\left(1 - \frac{s}{\beta_i}\right), \quad i = 1, 2. \tag{2.16}$$

From (1.4), we have

$$F_1 < F_2 \quad \text{when } \beta_1 = \beta_2 \quad \text{and} \quad \alpha_1 < \alpha_2 \tag{2.17}$$

and

$$F_1 < F_2 \quad \text{when } \alpha_1 = \alpha_2 > 1/t \quad \text{and} \quad \beta_1 < \beta_2. \tag{2.18}$$

Now let $D_{\phi_1}(s)=(-\infty,\beta_1)$, $D_{\phi_2}(s)=(-\infty,\beta_2)$ and $s_1=\beta_1$, then from (2.17) and (2.18), we have that $D_{\phi_1}(s)\subseteq D_{\phi_2}(s)$ and

$$\lim_{s\to\beta_1}\ \phi_1(s)=\alpha_1^{-\beta_1}\Gamma\left(1-\frac{\beta_1}{\beta_1}\right)=\Gamma(0+)=\infty, \tag{2.19}$$

see Abramowitz and Stegun (1965). □

On the other hand, when $\alpha_1=\alpha_2>\frac{1}{t}$ and $\beta_1<\beta_2$, we have

$$\lim_{s\to\beta_1}\ \phi_2(s)=\alpha_1^{-\beta_1}\Gamma\left(1-\frac{\beta_1}{\beta_2}\right)>0. \tag{2.20}$$

From (2.19) and (2.20), we have

$$\lim_{s\to\beta_1}\ [\phi_2(s)/\phi_1(s)]=0, \tag{2.21}$$

and hence the identifiability is proved. □

3. Estimation via EM Algorithm

In this section, we use the EM Algorithm to estimate the parameters of the pdf of the MTIWD given in (1.1) and (1.2). The EM Algorithm provides a simple computational method for fitting mixture models. The focus in this section is on the ML fitting of two IW mixture via the EM Algorithm. The essential nature of the algorithm is the alternation of expectation and maximization steps. (Refer to Maclachlan and Peel, 2000).

Concerning the E-step on the $(k+1)$th iteration, the updated estimate of the ith mixing proportion p_i is given by

$$p_i^{(k+1)}=\frac{1}{n}\sum_{j=1}^{n}\left[\frac{p_i^{(k)}\beta_i\alpha_i^{-\beta_i}y_j^{-(\beta_i+1)}\mathrm{e}^{-(\alpha_i y_j)^{-\beta_i}}}{\sum_{i=1}^{2}p_i^{(k)}\beta_i\alpha_i^{-\beta_i}y_j^{-(\beta_i+1)}\mathrm{e}^{-(\alpha_i y_j)^{-\beta_i}}}\right],\quad i=1,2. \tag{3.1}$$

In the M-step of the $(k+1)$th iteration, the updated estimates $\alpha_i^{(k+1)}$ and $\beta_i^{(k+1)}$ for $i=1,2$ are obtained, respectively, by solving the following systems of equations:

$$\alpha_i^{(k+1)}=\left[\frac{\sum_{j=1}^{n}\left\{\dfrac{p_i^{(k)}\beta_i\alpha_i^{-\beta_i}y_j^{-(\beta_i+1)}\mathrm{e}^{-(\alpha_i y_j)^{-\beta_i}}}{\sum_{i=1}^{2}p_i^{(k)}\beta_i\alpha_i^{-\beta_i}y_j^{-(\beta_i+1)}\mathrm{e}^{-(\alpha_i y_j)^{-\beta_i}}}\right\}}{\sum_{j=1}^{n}\left\{\dfrac{p_i^{(k)}\beta_i\alpha_i^{-\beta_i}y_j^{-(2\beta_i+1)}\mathrm{e}^{-(\alpha_i y_j)^{-\beta_i}}}{\sum_{i=1}^{2}p_i^{(k)}\beta_i\alpha_i^{-\beta_i}y_j^{-(\beta_i+1)}\mathrm{e}^{-(\alpha_i y_j)^{-\beta_i}}}\right\}}\right]^{-1/\beta_i} \tag{3.2}$$

and

$$\sum_{j=1}^{n}\left[s_i\left(\frac{1}{\beta_i}-(\log\alpha_i+\log(y_j))+(\alpha_i y_j)^{-\beta_i}\log(\alpha_i y_j)\right)\right]=0, \tag{3.3}$$

where

$$s_i=\frac{p_i^{(k)}\beta_i\alpha_i^{-\beta_i}y_j^{-(\beta_i+1)}\mathrm{e}^{-(\alpha_i y_j)^{-\beta_1}}}{\sum_{i=1}^{2}p_i^{(k)}\beta_i\alpha_i^{-\beta_i}y_j^{-(\beta_i+1)}\mathrm{e}^{-(\alpha_i y_j)^{-\beta_i}}},\quad i=1,2\quad\text{and}\quad p_2=1-p_1. \tag{3.4}$$

Note that α_i, and β_i in Eqs. (3.1), (3.2) and (3.4) should be raised to power k to indicate that they are the values obtained at the kth iteration, however, this has been suppressed for notational simplicity.

4. Simulation

In this section, we calculate the estimates of the five parameters p, α_1, α_2, β_1 and β_2 that appear in the pdf of the MTIWD given in (1.1) and (1.2) by using the EM Algorithm in a Monte Carlo simulation as follows:

1. Generate random samples of sizes $n = 25, 50, 75, 100$ for each choice of the vector of the parameters $\Theta = (p_1, \alpha_1, \alpha_2, \beta_1, \beta_2)$. Some of the choices of the parameters cover the unimodal model and the other choices cover the bimodal case.
2. The random samples of the mixtures are generated as follows:
   (a) Generate two uniform variates u_1 and u_2 from the Fortran numerical library (IMSL) using the routine DRNUN.
   (b) If $u_1 < p_1$, then use u_2 to generate a random variate t from the MTIWD by using (1.4) as $t = F_1^{-1}(u_2)$.
   (c) If $u_1 \geqslant p_1$, then use u_2 to generate a random variate t from the MTIWD by using (1.4) as $t = F_2^{-1}(u_2)$.

Table 2
Bias of the estimate of $\hat{\Theta}$ based on EM-Algorithm

$\Theta = (p_1, \alpha_1, \alpha_2, \beta_1, \beta_2)$	n	Bias				
		$\hat{p}_1$	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\beta}_1$	$\hat{\beta}_2$
0.2,1,2,2,3*	25	−0.0406	0.8364	−0.1628	0.5340	−0.4649
	50	−0.0420	0.8644	−0.1359	0.4871	−0.5130
	75	−0.0534	0.8652	−0.1333	0.4630	−0.5310
	100	−0.0198	0.8502	−0.1310	0.4228	−0.5422
0.3,1,2,2,3*	25	−0.2373	0.7613	−0.2389	0.3252	−0.6730
	50	−0.0887	0.7785	−0.2180	0.2689	−0.7240
	75	−0.0865	0.7944	−0.1935	0.2621	−0.7211
	100	−0.0474	0.7202	−0.2108	0.1605	−0.7190
0.5,1,2,2,3*	25	0.1032	0.5070	−0.4930	0.0427	−0.9573
	50	0.0112	0.5653	−0.4825	−0.0191	−1.0246
	75	0.0063	0.5725	−0.4585	−0.0183	−1.0144
	100	0.0046	0.5694	−0.4379	−0.0148	−1.0050
0.6,1,2,2,3*	25	−0.1041	0.5140	−0.4871	−0.0249	−1.0273
	50	0.0150	0.4650	−0.6225	−0.0902	−1.1700
	75	0.0081	0.4730	−0.5818	−0.0804	−1.1418
	100	0.0039	0.4853	−0.5728	−0.0746	−1.1324
0.2,2.5,1,2,2.9**	25	−0.2000	−1.2074	0.2949	−0.1834	−1.0791
	50	−0.1156	−1.2006	0.2994	−0.3001	−1.2001
	75	−0.0630	−1.1965	0.3035	−0.3309	−1.2309
	100	−0.0553	−1.1985	0.3015	−0.3335	−1.2335
0.3,2.5,1,2,2.9**	25	0.1718	−1.0664	0.4336	−0.3136	−1.2136
	50	−0.0406	−1.0548	0.4452	−0.3837	−1.2837
	75	0.0305	−1.0537	0.4463	−0.4057	−1.3057
	100	−0.0285	−1.0549	0.4451	−0.4201	−1.3201
0.5,2.5,1,2,2.9**	25	0.0926	−0.7767	0.7233	−0.3603	−1.2603
	50	−0.0694	−0.7705	0.7295	−0.4119	−1.3119
	75	0.0120	−0.7589	0.7411	−0.4206	−1.3206
	100	−0.0012	−0.7596	0.7404	−0.4260	−1.3260
0.6,2.5,1,2,2.9**	25	0.0601	−0.6178	0.8821	−0.3257	−1.2257
	50	0.0060	−0.6093	0.8907	−0.3962	−1.2692
	75	0.0008	−0.6113	0.8887	−0.3853	−1.2853
	100	0.0142	−0.6114	0.8886	−0.3868	−1.2868

*Unimodal; **bimodal.

Table 3
MSE of $\hat{\Theta}$ based on EM-Algorithm

$\Theta = (p_1, \alpha_1, \alpha_2, \beta_1, \beta_2)$	n	MSE				
		$\hat{p}_1$	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\beta}_1$	$\hat{\beta}_2$
0.2,1,2,2,3*	25	0.0020	0.7000	0.0265	0.2851	0.2000
	50	0.0020	0.7000	0.0185	0.2373	0.2000
	75	0.0020	0.7000	0.0178	0.2144	0.2000
	100	0.0004	0.7000	0.0172	0.1788	0.2000
0.3,1,2,2,3*	25	0.0563	0.6000	0.0571	0.1058	0.5000
	50	0.0079	0.6000	0.0475	0.0723	0.5000
	75	0.0075	0.6000	0.0400	0.0687	0.5000
	100	0.0022	0.5187	0.0400	0.0258	0.5000
0.5,1,2,2,3*	25	0.0107	0.3000	0.2430	0.0018	1.0000
	50	0.0001	0.3000	0.2328	0.0004	1.0000
	75	0.00004	0.3000	0.2103	0.0003	1.0000
	100	0.00002	0.3000	0.1918	0.0002	1.0000
0.6,1,2,2,3*	25	0.0108	0.2642	0.3000	0.0082	1.0000
	50	0.0002	0.2163	0.3000	0.0081	1.0000
	75	0.0001	0.2000	0.3000	0.0065	1.0000
	100	0.00001	0.2000	0.3000	0.0056	1.0000
0.2,2.5,1,2,2.9**	25	0.0400	1.0000	0.1000	0.1000	1.0000
	50	0.0134	1.0000	0.1000	0.1000	1.0000
	75	0.0040	1.0000	0.0900	0.1000	1.0000
	100	0.0031	1.0000	0.0900	0.1000	1.0000
0.3,2.5,1,2,2.9**	25	0.0295	1.1373	0.2000	0.1000	2.0000
	50	0.0016	1.1127	0.2000	0.1000	2.0000
	75	0.0009	1.1102	0.1992	0.1000	2.0000
	100	0.0008	1.0000	0.1982	0.1000	2.0000
0.5,2.5,1,2,2.9**	25	0.0086	0.6033	0.5000	0.2000	2.0000
	50	0.0048	0.5937	0.5000	0.2000	2.0000
	75	0.0001	0.5759	0.5000	0.1800	2.0000
	100	0.00001	0.5670	0.5000	0.1800	1.7582
0.6,2.5,1,2,2.9**	25	0.0036	0.3817	0.8000	0.1200	2.0000
	50	0.0004	0.3712	0.7933	0.1000	2.0000
	75	0.0001	0.3700	0.7897	0.1000	2.0000
	100	0.0001	0.3700	0.7896	0.1000	2.000

* Unimodal; ** bimodal.

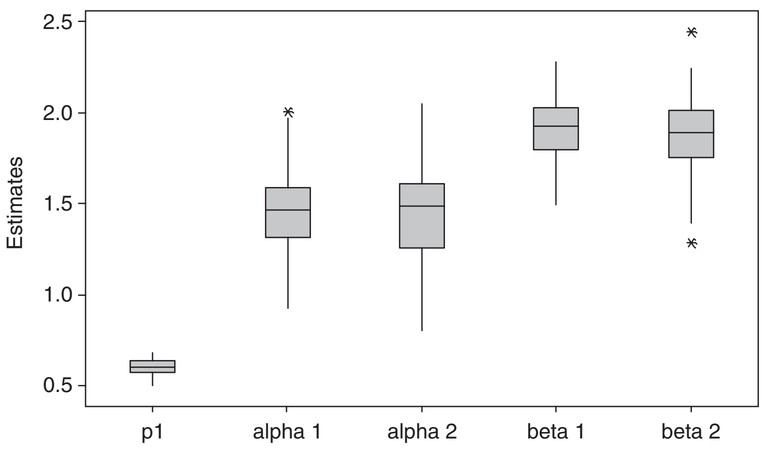

Fig. 5. Boxplot of the estimates.

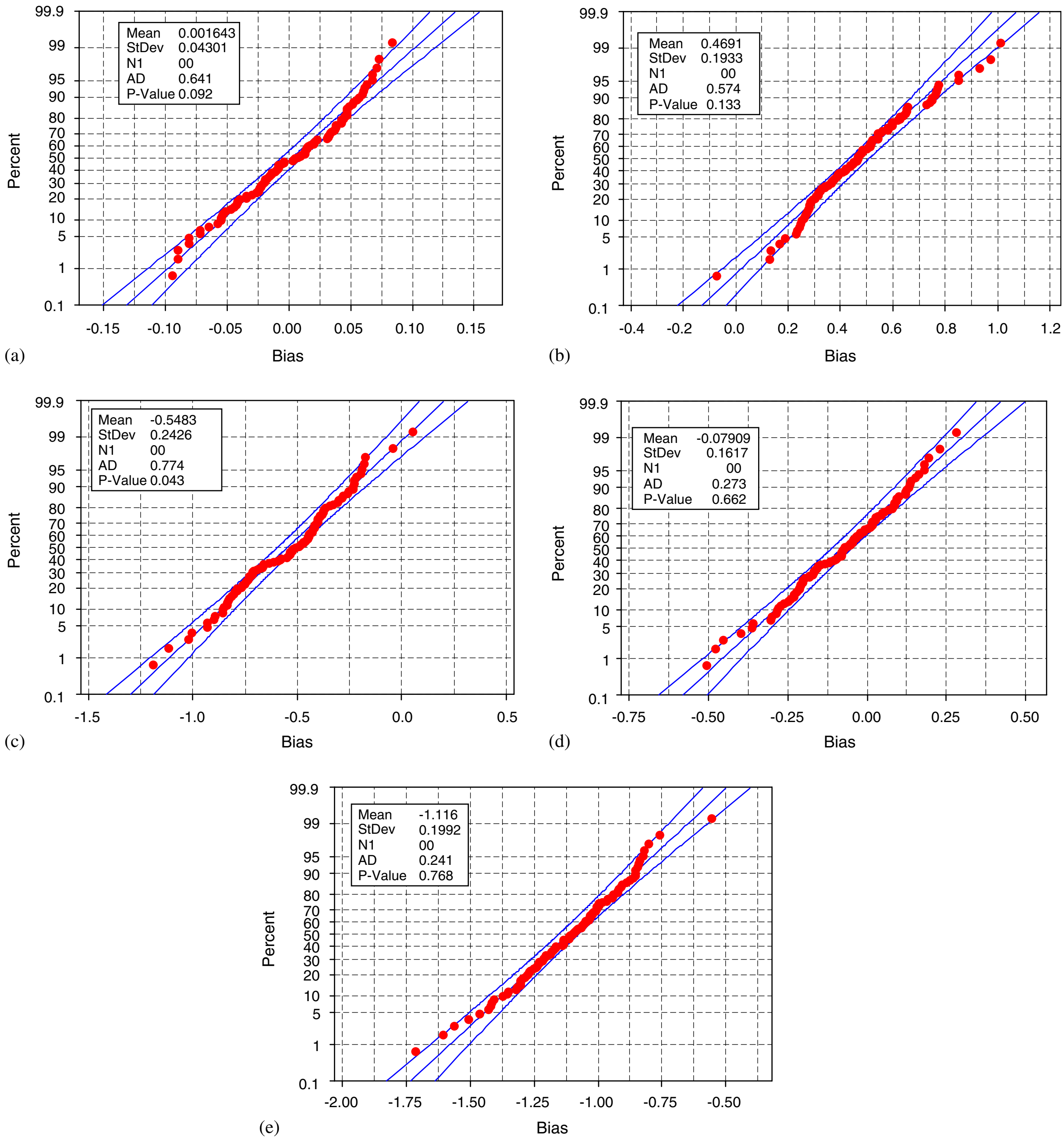

Fig. 6. (a) Probability plot for the bias of the estimate of (a) p1, (b) alpha1, (c) alpha2, (d) beta1, (e) beta2. Normal—95% CI.

3. The estimates of $p_1, \alpha_1, \alpha_2, \beta_1$ and β_2 are obtained by solving (3.1), (3.2) and (3.3). Eqs. (3.1) and (3.2) are written explicitly but Eq. (3.3) has to be solved numerically by using the subroutine DNEQNJ from the IMSL and random choices of the initial values.
4. The bias and the mean square errors of the estimates are calculated based on 10 000 Monte Carlo repetitions and the results are presented in Tables 2 and 3.
5. The EM Algorithm was terminated when $\log L(\Theta^{(k+1)}) - \log L(\Theta^{(k)})$ was less than $n \times 10^{-5}$, see Seidel et al. (2000).

Note that $L(\Theta^{(k+1)})$ and $L(\Theta^{(k)})$ denote the values of the likelihood function evaluated at the $(k+1)$th iteration and the kth iteration, respectively.

From Tables 2 and 3, we see that in most of the considered cases, the mean square errors of the estimated parameters decrease as n increases. The first 100 simulations of the estimates and their biases when $\Theta = (0.6, 1.0, 2.0, 2.0, 3.0)$ are plotted in Figs. 5 and 6. The boxplot in Fig. 5 shows that among 100 simulated estimates there is just one outlier for estimating α_1 and two outliers for estimating β_2. The probability plots in Figs. 6(a–e) show that the biases of estimates follow normal distributions.

5. Conclusion

In this paper, the behaviors of the mode and median of the MTIWD are investigated, based on different choices of the parameters. Also, the behaviors of the failure rate function are discussed through some different graphs. In addition, the identifiability property of the MTIWD is proved. Further, the estimation of the unknown parameters is obtained using the EM Algorithm. Finally, to investigate the performance of the estimation technique in the paper, a Monte Carlo simulation based on 10 000 runs is carried out.

Acknowledgments

The authors would like to thank the referees for their helpful comments, which improved the presentation of the paper.

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