

Comparison of estimates using record statistics from Weibull model: Bayesian and non-Bayesian approaches

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Abstract

This paper develops a Bayesian analysis in the context of record statistics values from the two-parameter Weibull distribution. The ML and the Bayes estimates based on record values are derived for the two unknown parameters and some survival time parameters e.g. reliability and hazard functions. The Bayes estimates are obtained based on a conjugate prior for the scale parameter and a discrete prior for the shape parameter of this model. This is done with respect to both symmetric loss function (squared error loss), and asymmetric loss function (linear-exponential (LINEX)) loss function. The maximum likelihood and the different Bayes estimates are compared via a Monte Carlo simulation study. A practical example consisting of real record values using the data from an accelerated test on insulating fluid reported by Nelson was used for illustration and comparison. Finally, Bayesian predictive density function, which is necessary to obtain bounds for predictive interval of future record is derived and discussed using a numerical example. The results may be of interest in a situation where only record values are stored.

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1. Introduction

The Weibull distribution is one of the most popular widely used models of failure time in life testing and reliability theory. The Weibull distribution has been shown to be useful for modeling and analysis of life time data in medical, biological and engineering sciences. Some applications of the Weibull distribution in forestry are given in Green et al. (1994). A great deal of research has been done on estimating the parameters of the Weibull distribution using both classical and Bayesian techniques, and a very good summary of this work can be found in Johnson et al. (1994). Recently, Hossain and Zimmer (2003) have discussed some comparisons of estimation methods for Weibull parameters using complete and censored samples.

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Record values and the associated statistics are of interest and important in many real life applications. In industry, many products fail under stress. For example, a wooden beam breaks when sufficient perpendicular force is applied to it, an electronic component ceases to function in an environment of too high temperature, and a battery dies under the stress of time. But the precise breaking stress or failure point varies even among identical items. Hence, in such experiments, measurements may be made sequentially and only the record values are observed. Thus, the number of measurements made is considerably smaller than the complete sample size. This “measurement saving” can be important when the measurements of these experiments are costly if the entire sample was destroyed. For more examples, see [Gulati and Padgett \(1994\)](#). There are also situations in which an observation is stored only if it is a record value. These include studies in meteorology, hydrology, seismology, athletic events and mining. In recent years, there has been much work on parametric and nonparametric inference based on record values. Among others are [Resnick \(1987\)](#), [Nagaraja \(1988\)](#), [Ahsanullah \(1993, 1995\)](#), [Arnold et al. \(1992, 1998\)](#), [Gulati and Padgett \(1994\)](#), [Raqab and Ahsanullah \(2001\)](#), and [Raqab \(2002\)](#). [Sultan and Balakrishnan \(1999\)](#) have developed some inferential method for the location and scale parameters of Weibull distribution. [Ahmadi and Arghami \(2001\)](#) discussed a comparison between the information (in Fisher’s sense) contained in a set of n upper record values with the Fisher information contained in n iid observations from the original distribution. They showed that, for the Weibull Model, the upper record values contain more information than the same number of iid observations.

In this paper, we obtain and compare several techniques of estimation based on record statistics for the two unknown parameter of the Weibull distribution. As well as the survival time parameters, namely the hazard and Reliability functions. Section 2 contains some preliminaries. In Section 3, the Bayes estimators of the parameters, the reliability and hazard functions are derived. This is done using the conjugate prior on the scale parameter and discretizing the shape parameter to a finite number of values. The Bayes estimates are obtained using both the symmetric loss function (*s.e.l.*) and the asymmetric loss function (Varian’s linear-exponential (LINEX)). A numerical example from an accelerated life-testing given by [Nelson \(1982\)](#) is used for illustration, and comparison is also given in Section 3. Results of a Monte Carlo simulation study conducted to evaluate the performance of these estimators compared to the MLEs in terms of mean squared error (MSE), are provided in Section 4. In Section 5, we provide Bayesian prediction interval for the future record with application examples. We conclude with some remarks and a brief summary of the results in Section 6.

2. Preliminaries

Let X_1, X_2, X_3, \dots a sequence of independent and identically distributed (iid) random variables with *cdf* $F(x)$ and *pdf* $f(x)$. Set $Y_n = \max(X_1, X_2, X_3, \dots, X_n)$, $n \geq 1$, we say that X_j is an upper record and denoted by $X_{U(j)}$ if $Y_j > Y_{j-1}$, $j > 1$. Assuming that $X_{U(1)}, X_{U(2)}, X_{U(3)}, \dots, X_{U(n)}$ are the first n upper record values arising from a sequence $\{X_i\}$ of iid Weibull variables with pdf

$$f(x) = \alpha\beta x^{\beta-1} \exp(-\alpha x^\beta), \quad x \geq 0, \alpha, \beta > 0, \quad (1)$$

and cdf

$$F(x) = 1 - \exp(-\alpha x^\beta), \quad x \geq 0, \alpha, \beta > 0, \quad (2)$$

where α and β are scale and shape parameters, respectively. This version of the Weibull distribution separates the two parameters and often simplifies the algebra in the subsequent Bayesian manipulations.

The reliability function $R(t)$, and the hazard (instantaneous failure rate) function $H(t)$ at mission time t for the Weibull distribution are given by

$$R(t) = \exp(-\alpha t^\beta), \quad t > 0, \quad (3)$$

and

$$H(t) = \alpha\beta t^{\beta-1}, \quad t > 0. \quad (4)$$

2.1. Maximum likelihood estimation (MLE)

The joint density function of the first n upper record values $x \equiv (x_{U(1)}, x_{U(2)}, \dots, x_{U(n)})$ is given by

$$f_{1,2,\dots,n}(x_{U(1)}, x_{U(2)}, \dots, x_{U(n)}) = f(x_{U(n)}) \prod_{i=1}^{n-1} \frac{f(x_{U(i)})}{1 - F(x_{U(i)})},$$

$$0 \leq x_{U(1)} < x_{U(2)} < \dots < x_{U(n)} < \infty, \tag{5}$$

where $f(\cdot)$, and $F(\cdot)$ are given, respectively, by (1) and (2) after replacing x by $x_{U(i)}$. The likelihood function based on the n upper record values x is given by

$$\ell(\alpha, \beta|x) = \alpha^n \beta^n u \exp(-\alpha x_{U(n)}^\beta), \quad u = \prod_{i=1}^n x_{U(i)}^{\beta-1}, \tag{6}$$

and the log-likelihood function may be written as

$$L(\alpha, \beta|x) \equiv \ln \ell = n \ln(\alpha\beta) - \alpha x_{U(n)}^\beta + (\beta - 1) \sum_{i=1}^n \ln(x_{U(i)}). \tag{7}$$

Assuming that the shape parameter β is known, the maximum likelihood estimator (MLE), $\hat{\alpha}_{ML}$ of the scale parameter α can be shown by using (7) to be

$$\hat{\alpha}_{ML} = n/x_{U(n)}^\beta. \tag{8}$$

If both of the parameters α and β are unknown, their MLEs, $\hat{\alpha}_{ML}$ and $\hat{\beta}_{ML}$, can be obtained by solving the following likelihood equations:

$$\frac{\partial L}{\partial \alpha} = 0 \quad \text{and} \quad \frac{\partial L}{\partial \beta} = 0. \tag{9}$$

By eliminating α between the two equations in (9), see Hoinkes and Padgett (1994), we obtain

$$\hat{\beta}_{ML} = \frac{n}{n \ln(x_{U(n)}) - \sum_{i=1}^n \ln(x_{U(i)})}, \tag{10}$$

and then

$$\hat{\alpha}_{ML} = \frac{n}{x_{U(n)}^{\hat{\beta}_{ML}}}. \tag{11}$$

The corresponding MLE's $\hat{R}_{ML}(t)$, and $\hat{H}_{ML}(t)$ of $R(t)$ and $H(t)$ are given by (3) and (4) after replacing α and β by $\hat{\alpha}_{ML}$ and $\hat{\beta}_{ML}$, respectively.

2.2. Loss function

For most statisticians, interested mainly in controlling the amount of variability, it has become standard practice to consider squared error loss function (*s.e.l.*). The symmetric nature of this function gives equal weight to overestimation as well as underestimation, while in the estimation of parameters of life time model, overestimation may be more serious than underestimation or vice-versa. For example, in the estimation of reliability and failure rate functions, an overestimate is usually much more serious than underestimate, in this case the use of symmetric loss function may be inappropriate as has been recognized by Basu and Ebrahimi (1991). This leads us to think that an asymmetrical loss function may be more appropriate.

A number of asymmetric loss functions are proposed for use, among these, one of the most popular asymmetric loss function is the linear-exponential loss function (LINEX). It has been introduced by Varian (1975), and various authors

as an example, Basu and Ebrahimi (1991) and Soliman (2002, 2005, 2006) have used this loss function in different estimation problems. This function rises approximately exponentially on one side of zero and approximately linearly on the other side. Under the assumption that the minimal loss occurs at $\phi^* = \phi$, the LINEX loss function for $\phi = \phi(\alpha, \beta)$ can be expressed as

$$L(\Delta) \propto \exp(c\Delta) - c\Delta - 1, \quad c \neq 0. \tag{12}$$

where $\Delta = (\phi^* - \phi)$, ϕ^* is an estimate of ϕ . The sign and magnitude of the shape parameter c represents the direction and degree of symmetry, respectively. (If $c > 0$, the overestimation is more serious than underestimation, and vice-versa.) For c close to zero, the LINEX loss is approximately s.e.l. and therefore almost symmetric.

The posterior expectation of the LINEX loss function (12) is

$$E_\phi [L(\phi^* - \phi)] \propto \exp(c\phi^*) E_\phi[\exp(-c\phi)] - c(\phi^* - E_\phi(\phi)) - 1, \tag{13}$$

where $E_\phi(\cdot)$ denotes the posterior expectation with respect to the posterior density of ϕ . The Bayes estimator of ϕ , denoted by ϕ_{BL}^* under the LINEX loss function is the value ϕ^* which minimizes (13). It is

$$\phi_{BL}^* = -\frac{1}{c} \ln \{E_\phi[\exp(-c\phi)]\}, \tag{14}$$

provided that the expectation $E_\phi[\exp(-c\phi)]$ exists and is finite. The problem of choosing the value of the parameter c is discussed in Calabria and Pulcini (1996).

3. Bayes estimation

In this section, we estimate α , β , $R(t)$ and $H(t)$, by considering both symmetric (squared error) loss function, and asymmetric (LINEX) loss function.

3.1. Known shape parameter β

Under the assumption that the shape parameter β is known, we assume a gamma $\gamma(a, b)$ conjugate prior for α as

$$\pi(\alpha) = b^a \alpha^{a-1} \exp[-b\alpha] / \Gamma(a), \quad \alpha > 0, \quad a, b > 0. \tag{15}$$

Combining the likelihood function (6) and the prior density (15), we obtain the posterior density of α in the form

$$\pi^*(\alpha|x) = v^{(n+a)} \alpha^{(n+a-1)} \exp[-\alpha v] / \Gamma(n+a), \quad v = \left(b + x_{U(n)}^\beta\right). \tag{16}$$

3.1.1. Bayes estimator based on squared error loss function

Under a s.e.l. function, the Bayes estimator $\tilde{\phi}_{BS}$ of a function $\phi(\alpha)$ is the posterior mean. Hence, the Bayes estimator $\tilde{\alpha}_{BS}$ of α is then given by

$$\tilde{\alpha}_{BS} = E(\alpha|x) = \int_0^\infty \alpha \pi^*(\alpha|x) d\alpha = \frac{n+a}{v}. \tag{17}$$

Similarly, in making use of (3) and (16), the Bayes estimator for the reliability function $R(t)$ is given by

$$\tilde{R}_{BS}(t) = \int_0^\infty \exp(-\alpha t^\beta) \pi^*(\alpha|x) d\alpha = \left(1 + \frac{t^\beta}{v}\right)^{-(n+a)}, \tag{18}$$

and from (4) and (17), the Bayes estimator for the hazard function $H(t)$ is given by

$$\tilde{H}_{BS}(t) = \beta t^{\beta-1} \left(\frac{n+a}{v}\right). \tag{19}$$

3.1.2. Bayes estimator based on LINEX loss function

Under the LINEX loss function, and by using (14), the Bayes estimator $\tilde{\alpha}_{BL}$ for α , is given by

$$\tilde{\alpha}_{BL} = -\frac{1}{c} \ln \left[\int_0^\infty \{\exp(-c\alpha)\} \pi^*(\alpha|x) d\alpha \right], \tag{20}$$

By using (16), the above equation simplifies to

$$\tilde{\alpha}_{BL} = \frac{n+a}{c} \ln \left(1 + \frac{c}{v} \right). \tag{21}$$

Similarly, the Bayes estimators for $R(t)$ and $H(t)$ are given, respectively, by

$$\tilde{R}_{BL}(t) = -\frac{1}{c} \ln \left[\sum_{i=0}^\infty \frac{(-c)^i}{i!} \left(1 + \frac{it^\beta}{v} \right)^{-(n+a)} \right], \tag{22}$$

and

$$\tilde{H}_{BL}(t) = \frac{n+a}{c} \ln \left(1 + \frac{c\beta t^{\beta-1}}{v} \right). \tag{23}$$

3.2. Unknown scale and shape parameters α and β

It is well known that, for the Bayes estimators, the performance depends on the form of the prior distribution and the loss function assumed. Under the assumption that both the parameters α and β are unknown, no analogous reduction via sufficiency is possible for the likelihood corresponding to a sample of records from the Weibull density (1). Also, specifying a general joint prior for α and β leads to computational complexities. In trying to solve this problem and simplify the Bayesian analysis, we use Soland’s method. Soland (1969) considered a family of joint prior distributions that places continuous distributions on the scale parameter and discrete distributions on the shape parameter.

We assume that the shape parameter β is restricted to a finite number of values $\beta_1, \beta_2, \dots, \beta_k$ with respective prior probabilities $\eta_1, \eta_2, \dots, \eta_k$ such that $0 \leq \eta_j \leq 1$, and $\sum_{j=1}^k \eta_j = 1$. [i.e. $\Pr(\beta = \beta_j) = \eta_j$]. Further, suppose that conditional upon $\beta = \beta_j$, α has a natural conjugate prior with distribution having a gamma (a_j, b_j) with density

$$\pi(\alpha|\beta = \beta_j) = \frac{b_j^{a_j} \alpha^{a_j-1}}{\Gamma(a_j)} \exp[-b_j\alpha], \quad a_j, b_j, \alpha > 0, \tag{24}$$

where a_j and b_j are chosen so as to reflect prior beliefs on α given that $\beta = \beta_j$. Then given the set of the first n upper record values x , the conditional posterior pdf of α is given by

$$\pi^*(\alpha|\beta = \beta_j, x) = \frac{B_j^{A_j} \alpha^{A_j-1}}{\Gamma(A_j)} \exp[-B_j\alpha], \quad A_j, B_j; \alpha > 0, \tag{25}$$

which is a gamma (A_j, B_j) , where

$$A_j = a_j + n \quad \text{and} \quad B_j = b_j + x_{U(n)}^{\beta_j}. \tag{26}$$

On applying the discrete version of Bayes’ theorem, the marginal posterior probability distribution of β_j is given by

$$\begin{aligned} P_j &= \Pr(\beta = \beta_j|x) = A \int_0^\infty \frac{\eta_j b_j^{a_j} \alpha^{n+a_j-1} \rho_j^n u_j}{\Gamma(a_j)} \exp[-\alpha \{b_j + x_{U(n)}^{\beta_j}\}] d\alpha. \\ &= A \frac{\eta_j b_j^{a_j} \rho_j^n u_j \Gamma(A_j)}{B_j^{A_j} \Gamma(a_j)}, \end{aligned} \tag{27}$$

where $u_j = \prod_{i=1}^n x_{U(i)}^{\beta_j - 1}$ and A is a normalized constant given by

$$A^{-1} = \sum_{j=1}^k \frac{\eta_j b_j^{a_j} \beta_j^n u_j \Gamma(A_j)}{B_j^{A_j} \Gamma(a_j)}.$$

3.2.1. Estimators based on squared error loss function(s.e.l)

The Bayes estimators $\tilde{\alpha}_{BS}, \tilde{\beta}_{BS}$ of the parameters α and β , under the s.e.l function are obtained by using the posterior pdfs (25) and (27).

The Bayes estimator for the scale parameter α is given by

$$\tilde{\alpha}_{BS} = \int_0^\infty \sum_{j=1}^k P_j \alpha \pi^*(\alpha | \beta = \beta_j, x) d\alpha.$$

which upon using (25), simplifies to

$$\tilde{\alpha}_{BS} = \sum_{j=1}^k (P_j A_j) / B_j. \tag{28}$$

The Bayes estimator of β is given by

$$\tilde{\beta}_{BS} = \sum_{j=1}^k P_j \beta_j. \tag{29}$$

Similarly, the Bayes estimators $\tilde{R}(t)_{BS}$ and $\tilde{H}(t)_{BS}$ of the reliability and hazard functions $R(t)$ and $H(t)$ are given, respectively, by

$$\tilde{R}(t)_{BS} = \int_0^\infty \sum_{j=1}^k P_j \exp[-\alpha t^{\beta_j}] \pi^*(\alpha | \beta = \beta_j, x) d\alpha = \sum_{j=1}^k P_j \left(\frac{B_j}{D_j}\right)^{A_j}, \tag{30}$$

and

$$\tilde{H}(t)_{BS} = \sum_{j=1}^k P_j \left(\frac{\beta_j A_j t^{\beta_j - 1}}{B_j}\right), \tag{31}$$

where $D_j = B_j + t^{\beta_j}$, A_j and B_j are as given in (26).

3.2.2. Estimators based on LINEX loss function

Under the LINEX loss function (13), the Bayes estimator $\tilde{\phi}_{BL}$ of a function $\phi(\alpha, \beta)$ is given by (14). The Bayes estimator for the scale parameter α is given by

$$\tilde{\alpha}_{BL} = -\frac{1}{c} \ln \left[\int_0^\infty \sum_{j=1}^k P_j \exp[-\alpha c] \pi^*(\alpha | \beta = \beta_j, x) d\alpha \right] = -\frac{1}{c} \ln \left[\sum_{j=1}^k P_j \left(1 + \frac{c}{B_j}\right)^{-A_j} \right] \tag{32}$$

and the Bayes estimator for β is given by

$$\tilde{\beta}_{BL} = -\frac{1}{c} \ln \left[\sum_{j=1}^k P_j \exp\{-c\beta_j\} \right]. \tag{33}$$

Similarly, the Bayes estimator for the reliability function $R(t)$ is given by

$$\tilde{R}(t)_{BL} = -\frac{1}{c} \ln \left[\int_0^\infty \sum_{j=1}^k P_j \exp[-cR(t)] \pi^*(\alpha | \beta = \beta_j, x) \, d\alpha \right],$$

where $R(t)$ is given in (3). By using the exponential series, after some simplification we obtain

$$\tilde{R}(t)_{BL} = -\frac{1}{c} \ln \left[\sum_{j=1}^k \sum_{i=0}^\infty P_j \frac{(-c)^i}{i!} \left(1 + \frac{it\beta_j}{B_j} \right)^{-A_j} \right], \tag{34}$$

and the Bayes estimator for the hazard function $H(t)$ is obtained as

$$\tilde{H}(t)_{BL} = -\frac{1}{c} \ln \left[\sum_{j=1}^k P_j \left(1 + \frac{c\psi_j(t)}{B_j} \right)^{-A_j} \right], \quad \psi_j(t) = \beta_j t^{\beta_j - 1}. \tag{35}$$

To implement the calculations in this section, it is first necessary to elicit the values of (β_j, η_j) and the hyperparameters (a_j, b_j) in the conjugate prior (24), for $j = 1, 2, \dots, k$. The former pairs of values are fairly straightforward to specify, but for (a_j, b_j) it is necessary to condition prior beliefs about α on each β_j in turn, and this can be difficult in practice. An alternative method for obtaining the values (a_j, b_j) can be based on the expected value of the reliability function $R(t)$ conditional on $\beta = \beta_j$, which is given using (24) by

$$E_{\alpha|\beta_j} [R(t) | \beta = \beta_j] = \int e^{-\alpha t^{\beta_j}} \frac{b_j^{a_j} \alpha^{a_j-1}}{\Gamma(a_j)} e^{-\alpha b_j} \, d\alpha = \left(1 + \frac{t^{\beta_j}}{b_j} \right)^{-a_j}. \tag{36}$$

Now, suppose that prior beliefs about the lifetime distribution enable one to specify two values $(R(t_1), t_1), (R(t_2), t_2)$. Thus, for these two prior values $R(t = t_1)$ and $R(t = t_2)$, the values of a_j and b_j for each value β_j , can be obtained numerically from (36). If there are no prior beliefs, a nonparametric procedure can be used to estimate the corresponding two different values of $R(t)$, see Martz and Waller (1982, p. 105).

Example 3.1. To illustrate the estimation techniques developed in this section, we consider the real data set which was also used in Lawless (1982, p. 185). These data are from Nelson (1982), concerning the data on time to breakdown of an insulating fluid between electrodes at a voltage of 34 kV (minutes). The 19 times to breakdown are

0.96, 4.15, 0.19, 0.78, 8.01, 31.75, 7.35, 6.50, 8.27, 33.91, 32.52, 3.16, 4.85,
2.78, 4.67, 1.31, 12.06, 36.71, 72.89.

Therefore, we observe the following seven upper record values:

0.96, 4.15, 8.01, 31.75, 33.91, 36.71, 72.89.

A model suggested by engineering considerations is that, for a fixed voltage level, time to breakdown has a Weibull distribution. The computations in this example are done using gamma prior for the scale parameter and a discrete prior for the shape parameter. The hyperparameters of the gamma prior (24) and the values of β_j are derived by the following steps:

1. based on the above seven upper records, we estimate two values of the reliability function using a nonparametric procedure $R(t_i = X_{U(i)}) = (n - i + 0.625)/(n + 0.25)$, $i = 1, 2, \dots, n$ with $(n = 7)$, see Martz and Waller (1982). So, we assume that the reliability for times $t_1 = 4.15$, and $t_2 = 31.75$ are, respectively, $R(t_1) = 0.776$, and $R(t_2) = 0.50$;
2. concerning the value of the MLE of the parameter β from (10), $(\beta_{ML} = 0.599)$, we assume that β_j takes the values: $0.5(0.05)0.95$;

Table 1
Prior information, Hyper parameter values and the posterior probabilities

j	1	2	3	4	5
β_j	0.50	0.55	0.60	0.65	0.70
η_j	0.1	0.1	0.1	0.1	0.1
a_j	17.907	2.001	1.091	0.763	0.593
b_j	142.828	16.193	8.976	6.393	5.071
u_j	0	0	0	0.002	0.004
P_j	0.002	0.031	0.090	0.137	0.159
j	6	7	8	9	10
β_j	0.75	0.80	0.85	0.90	0.95
η_j	0.1	0.1	0.1	0.1	0.1
a_j	0.488	0.418	0.366	0.327	0.297
b_j	4.271	3.737	3.358	3.077	2.860
u_j	0.010	0.026	0.064	0.160	0.4
P_j	0.156	0.140	0.118	0.094	0.073

Table 2
Estimates of α , β , $R(t)$ and $H(t)$ with $t = 5$

	$(\cdot)_{ML}$	$(\cdot)_{BS}$	$(\cdot)_{BL}$		
			$c = -1$	$c = -2$	$c = 2$
α	0.536	0.252	0.260	0.268	0.238
β	0.599	0.756	0.762	0.767	0.744
$R(t)$	0.2451	0.465	0.474	0.484	0.446
$H(t)$	0.1685	0.121	0.122	0.123	0.119

3. the two prior values obtained in step 1 are substituted into (36), where a_j and b_j are solved numerically for each given β_j , $j = 1, 2, \dots, 10$, using the Newton–Raphson method.

Table 1 gives the values of the hyperparameters and the posterior probabilities derived for each β_j . The ML estimates $(\cdot)_{ML}$, and the Bayes estimates $((\cdot)_{BS}, (\cdot)_{BL})$ of α , β , $R(t)$, and $H(t)$, are computed and the results are displayed in Table 2.

It should be noted that the ML estimates for the parameters α and β using the same set of data, (complete sample ($n = 19$)) are obtained (see Lawless, 1982, p. 187) to be $\tilde{\alpha} = 0.145$ and $\tilde{\beta} = 0.771$. We conclude that the proposed Bayes estimates based on record values are better than the corresponding ML estimates.

4. Simulation study and comparisons

Since we are not aware of any estimation results concerning Bayes estimation based on record values, a simulation study was conducted in order to compare the performance of the presented Bayes estimators with the known non-Bayes estimator such as the maximum likelihood estimator. We use the inverse sampling scheme, where the number of records generated from sampling with random sample size.

It is well known that the two-parameter Weibull distribution can represent decreasing, constant or increasing failure rates. These corresponding to the three sections of the “bathtub curve” of reliability, referred to also as “burn-in,” “random,” and “wear-out,” phases of life. So, the simulation were carried out for different choices of the value of the shape parameter β in each case, corresponding to decreasing, constant and increasing failure rates. Since α is a scale parameter, only one value of α needs to be considered, because changing the value of α is equivalent to multiplying the sample values by a constant.

Table 3
MSEs of the estimates of α , $R(t)$ and $H(t)$ when $t = 0.5$

n	$\hat{\alpha}_{ML}$	$\tilde{\alpha}_{BS}$	$\tilde{\alpha}_{BL}$	\hat{R}_{ML}	\tilde{R}_{BS}	\tilde{R}_{BL}	\hat{H}_{ML}	\tilde{H}_{BS}	\tilde{H}_{BL}
	$c = -2$			$c = 00$			$c = -1$		
3	4.641	0.259	0.153	0.023	0.003	0.003	2.611	0.146	0.111
5	1.637	0.148	0.0413	0.011	0.002	0.002	0.921	0.083	0.069
7	0.869	0.075	0.075	0.007	0.002	0.002	0.489	0.042	0.040

4.1. The case of known β

When shape parameter β is known, we use a numerical technique to compare the ML and Bayes estimates according the following steps:

1. For given values ($a = 2, b = 3$), generate $\alpha = 1.514$ from the prior pdf (15).
2. Using the value $\alpha = 1.514$ from step 1, with $\beta(\text{known})=3$, we generate $n, (n = 3, 5, 7)$ upper record values from the Weibull pdf in (1). The Weibull records are generated using the transformation: $X_i = (-\ln U_i/\alpha)^{1/\beta}$, where U_i is the uniformly distributed random variate.
3. The ML estimate $\hat{\alpha}_{ML}$ of α is calculated using (11), and the corresponding ML estimates \hat{R}_{ML} and \hat{H}_{ML} are then computed according to (3) and (4), respectively, at time t (chosen to be 0.5), after replacing α by the corresponding MLE $\hat{\alpha}_{ML}$ (with β known).
4. The different Bayes estimates of $\alpha, R(t)$, and $H(t)$ are computed through (17)–(23).
5. Steps 1–4 are repeated 10,000 times, and the mean square error (MSE) for each method was calculated. The results are displayed in Table 3 for different choices of the parameter c (the shape parameter of the LINEX loss function (13)).

Note: Experimentation with different values of the prior parameters a and b led to the same results, which confirms convergence of the sampler and shows a good stability with respect to the prior settings.

4.2. The case of unknown α and β

As indicated in Section 3.2, in the case of unknown α and β , specifying a general joint prior for α and β leads to computational complexities. Soland’s method provides a simpler alternative which can serve to give good approximations to the corresponding more general case of assuming continuous priors for α and for α given β . For this case, one sample does not tell us much. A simulation study was conducted in order to compare the ML and Bayes estimation methods according to the following steps:

1. Samples of upper record values with size $n, (n = 3, 5, 7)$ were generated from the standard Weibull distribution ($\alpha = 1, \beta$), with $\beta = 0.5, 1, 3$. corresponding to decreasing, constant and increasing failure rates, respectively. As mentioned in example 3.1. we assume the prior for the parameter β as follows:
 - (a) for $\beta = 0.5$, we approximate the prior over the interval (0.1, 1) by the discrete prior with β taking the 10 values $0.1(0.1)1$, each with probability 0.1.
 - (b) for $\beta = 1$ and $\beta = 3$, we consider a discrete prior over the intervals (0.4, 1.5) and (2.5, 3.5), respectively.
2. A nonparametric procedure as described in Example 3.1 was used to obtain the values of the hyperparameters a_j and b_j for a given values of $\beta_j, j = 1, 2, \dots, 10$.
3. The MLEs $(\cdot)_{ML}$ and the Bayes estimates $(\cdot)_{BS}, (\cdot)_{BL}$ of $\alpha, \beta, R(t)$, and $H(t)$ are computed by using the results in Section 3.2.
4. The above steps are repeated 10,000 times to evaluate the MSEs of the ML and Bayes estimates, and the results are presented in Tables 4 and 5 below:

Table 4
MSEs of the estimates of α and β

n	$\hat{\alpha}_{ML}$	$\tilde{\alpha}_{BS}$	$\tilde{\alpha}_{BL}$	$\hat{\beta}_{ML}$	$\tilde{\beta}_{BS}$	$\tilde{\beta}_{BL}$
<i>Simulated records with $(\alpha = 1, \beta = 0.5)$</i>						
			$c = 2$			$c = 2$
3	3.7×10^9	4.338	0.331	10.481	0.044	0.033
5	1.762	0.427	0.232	0.446	0.035	0.026
7	0.655	0.239	0.188	0.149	0.029	0.022
<i>Simulated records with $(\alpha = 1, \beta = 1)$</i>						
			$c = 2$			$c = -1$
3	8.8×10^{18}	2.624	0.172	27.217	0.016	0.013
5	0.831	0.551	0.148	1.847	0.015	0.012
7	0.629	0.405	0.125	0.592	0.014	0.010
<i>Simulated records with $(\alpha = 1, \beta = 3)$</i>						
			$c = 1.5$			$c = -1.5$
3	1.4×10^{71}	0.929	0.204	299.7	0.009	0.005
5	0.927	0.457	0.172	17.03	0.008	0.004
7	0.583	0.284	0.130	5.269	0.005	0.003

Table 5
MSEs of the estimates of $R(t)$ and $H(t)$ with $t = 0.5$

n	\hat{R}_{ML}	\tilde{R}_{BS}	\tilde{R}_{BL}	\hat{H}_{ML}	\tilde{H}_{BS}	\tilde{H}_{BL}
<i>Simulated records with $(\alpha = 1, \beta = 0.5)$</i>						
			$c = 2$			$c = 2$
3	0.121	0.043	0.039	4.4×10^{15}	3.873	0.235
5	0.094	0.034	0.027	1.418	0.312	0.125
7	0.077	0.027	0.022	0.255	0.141	0.085
<i>Simulated records with $(\alpha = 1, \beta = 1)$</i>						
			$c = -2$			$c = 1$
3	0.092	0.026	0.022	1.3×10^6	2.694	0.318
5	0.074	0.020	0.016	0.708	0.487	0.219
7	0.064	0.019	0.015	0.487	0.317	0.171
<i>Simulated records with $(\alpha = 1, \beta = 3)$</i>						
			$c = -2$			$c = 2$
3	0.022	0.008	0.006	95.86	0.515	0.117
5	0.021	0.005	0.004	0.642	0.268	0.097
7	0.020	0.004	0.003	0.575	0.178	0.072

5. Prediction of the future records

In the context of prediction of the future record observations, the prediction intervals provide bounds to contain the results of a future record, based upon the results of the previous record observed from the same sample. This section is devoted for deriving the Bayes predictive density function, which is necessary to obtain bounds for predictive interval. Suppose that we observe only the first n upper record observations $x \equiv (x_{U(1)}, x_{U(2)}, \dots, x_{U(n)})$, and the goal is to obtain the Bayes predictive interval for the s th future upper record, where $1 \leq n < s$. Let $Y \equiv X_{u(s)}$ be the s th upper record value. The conditional density function of Y for given $x_n = x_{U(n)}$ is given, (Ahsanullah, 1995), by

$$f(y|x_n; \theta) = \frac{[w(y) - w(x_n)]^{s-n-1}}{\Gamma(s-n)} \frac{f(y)}{1 - F(x_n)}, \tag{37}$$

where $w(\cdot) = -\ln[1 - F(\cdot)]$.

Upon using the Weibull distribution, with pdf given by (1), the conditional density function (37) is given by

$$f(y|x_n; \alpha, \beta) = \frac{\beta y^{\beta-1} [\alpha \gamma(y)]^{s-n-1}}{\alpha \Gamma(s-n)} \exp[-\alpha \gamma(y)], \quad \gamma(y) = (y^\beta - x_n^\beta). \tag{38}$$

The Bayes predictive density function of y given the observed record x is given by

$$f(y|x) = \int_{\theta} f(y|x_n; \alpha, \beta) \sum_{j=1}^k P_j \pi^*(\alpha|\beta = \beta_j, x) d\alpha. \tag{39}$$

Substituting from (38), (25) and (27) into (39), we get

$$f(y|x) = \sum_{j=1}^k \frac{\beta_j P_j B_j^{A_j}}{Bet(n + a_j, s - n)} y^{\beta_j-1} (\gamma_j(y))^{s-n-1} (w_j(y))^{-s-a_j}, \tag{40}$$

where A_j and B_j are as given by (26), $Bet(\cdot, \cdot)$ is the beta function of the second kind, and

$$\gamma_j(y) = (y^{\beta_j} - x_n^{\beta_j}) \quad \text{and} \quad w_j(y) = b_j + y^{\beta_j}. \tag{41}$$

It follows that the lower and upper 100τ% prediction bounds for $Y = X_{U(s)}$, given the past record values x , can be derived using the predictive survival function defined by

$$\begin{aligned} f(Y \geq \lambda|x) &= \int_{\lambda}^{\infty} f(y|x) dy \\ &= \sum_{j=1}^k \frac{P_j}{Bet(n + a_j, s - n)} InBet(n + a_j, s - n; \delta_j), \end{aligned} \tag{42}$$

where $\delta_j = (\lambda^{\beta_j} - x_n^{\beta_j}) / B_j$, and $InBet(z_1, z_2, \xi)$ is the incomplete beta function defined by

$$InBet(z_1, z_2, \xi) = \int_{\xi}^{\infty} t^{z_1-1} / (1+t)^{z_1+z_2} dt.$$

Iterative numerical methods are required to obtain the lower and upper 100τ% prediction bounds for $Y = X_{U(s)}$ by finding λ from Eq. (42), using

$$\Pr[LL(x) < Y < UL(x)] = \tau,$$

where $LL(x)$ and $UL(x)$ are the lower and upper limits, respectively, satisfying

$$\Pr[Y > LL(x)|x] = (1 + \tau)/2 \quad \text{and} \quad \Pr[Y > UL(x)|x] = (1 - \tau)/2, \tag{43}$$

It is often important to predict the first unobserved record value $X_{U(n+1)}$; the predictive survival function for $Y_{n+1} = X_{U(n+1)}$ is given from (42) by setting $s = n + 1$ as

$$f(Y_{n+1} \geq \lambda|x) = \sum_{j=1}^k P_j (1 + \delta_j)^{-(n+a_j)}. \tag{44}$$

Iterative numerical methods are also required to obtain prediction bounds for Y_{n+1} .

Example 5.1. The above prediction procedure is demonstrated by using simulated sets of record values from the Weibull model (1). Samples of upper record values of size $n = 3, 5, 7$ are simulated from the Weibull distribution with $(\alpha, \beta) = (1, 1), (1, 2), (1, 3)$, which include the exponential, Rayleigh, and Weibull models, respectively. Using our

Table 6
The lower (*LL*), the upper (*UL*) and the width of the 95% prediction intervals for the future upper record $X_{U(n+1)}$ ($n = 3, 5, 7$)

n	(α, β)	Previous record values	<i>LL</i>	<i>UL</i>	Width
3	(1,1)	0.004, 1.489, 3.440	3.46	7.80	4.34
5		0.429, 0.560, 1.519, 2.951, 3.812	3.83	6.57	2.74
7		0.325, 2.295, 2.517, 2.597, 2.693, 2.984, 5.325	5.34	7.90	2.56
3	(1,2)	2.279, 2.365, 2.565	2.57	4.41	1.84
5		0.729, 0.797, 1.354, 1.426, 2.280	2.29	3.26	0.97
7		0.405, 1.413, 1.604, 1.695, 2.095, 2.152, 2.662	2.67	3.44	0.77
3	(1,3)	1.378, 1.527, 1.680	1.68	2.43	0.75
5		1.087, 1.233, 1.589, 1.833, 1.8659	1.87	2.36	0.49
7		1.338, 1.577, 1.617, 1.683, 1.723, 1.783, 1.885	1.90	2.24	0.34

results in Eq. (44), the lower and the upper 95% prediction bounds for the next record values $X_{U(n+1)}$, for the three cases ($n = 3, 5, 7$) are obtained and displayed in Table 6.

Example 5.2. Based on the seven record values from Example 5.1, with the corresponding hyperparameter values obtained in the same example in Table 4, and using the results in (43) and (44), the lower and the upper 95% prediction bounds for the next record values $X_{U(8)}$ are, respectively, 73.28 and 170.79.

6. Conclusion

Based on the set of the upper record values, the present paper proposes classical and Bayesian approaches to estimate the two unknown parameters as well as the reliability and hazard functions for the Weibull model. We also considered the problem of predicting future record in a Bayesian setting. The Bayes estimators are obtained using both symmetric and asymmetric loss functions. It appears from this study that the Bayes method of estimation based on record statistics is superior to the ML method. Comparisons are made between the different estimators based on a simulation study and practical example using a set of real record values. The effect of symmetric and asymmetric loss functions was examined and the following were observed:

1. Table 3 (β is known) shows that the Bayes estimates relative to the LINEX loss function has the smallest (MSE) as compared with both quadratic Bayes estimates or the MLEs. Also, the MSEs decrease as n increases. If the experimenter has proper prior knowledge about α the Bayes estimations are closer to the true value of α .
2. For the case of the unknown shape and scale parameters, the use of a discrete distribution for the shape parameter resulted in a closed form expression for the posterior pdf. The equal probabilities chosen in the discrete distributions caused an element of uncertainty, which can be desirable in some cases.
3. Tables 4 and 5 show that the Bayes estimates based on symmetric and asymmetric loss functions perform better than the MLEs, and the asymmetric Bayes estimates are sensitive to the values of the shape parameter c of the LINEX loss function. For small $|c|$, the Bayes estimators under the LINEX loss function are very close to the estimators under the s.e.l. function. This is one of the useful properties of working with the LINEX loss function.
4. For estimating α and β in the case of small record sample size, it is recommended that one uses the Bayes method of estimation. Note that (see Tables 4 and 5) when the sample size is small ($n = 3$), the ML method has a large MSE.
5. In general, from Tables 4 and 5, we note that the MSE for all Bayes methods decreases with increasing β , while the MSE for ML method increases with increasing β .
6. The analytical case with which results can be obtained using asymmetric loss functions makes them attractive for use in applied problems and in assessing the effects of departures from assumed symmetric loss functions.

7. It is clear that the variance of the Weibull(α, β) distribution tends to zero as the shape parameter β tends to infinity. This implies that as β gets larger, the observations concentrate on a shorter domain. It then follows that the width of the predictive interval decrease as β increase, see Tables 6.

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