12.1 Functions of several variables

Definition1

*A **function of two variables** is a rule that assigns a real number $f(x,y)$ to each ordered pair of real numbers (x,y) in the domain of the function.*
For a function f defined on the domain $D \subseteq \mathbb{R}^2$, we sometimes write $f: D \subseteq \mathbb{R}^2 \to \mathbb{R}$ to indicate that f maps points in two dimensions to real numbers.

*Likewise, a **function of three variables** is a rule that assigns a real number $f(x,y,z)$ to each ordered triple of real numbers (x,y,z) in the domain $D \subseteq \mathbb{R}^3$ of the function. We sometimes write $f: D \subseteq \mathbb{R}^3 \to \mathbb{R}$ to indicate that f maps points in three dimensions to real numbers.*

For instance, $f(x,y,z) = \dfrac{\cos(x+z)}{xy}$ and $g(x,y,z) = x^2 y - e^{xz}$ are both functions of the three variables x, y and z .

Example 1 (Finding the Domain of a Function of Two Variables)

Find and sketch the domain for
(a) $f(x,y) = x \ln y$.
(b) $g(x,y) = \dfrac{2x}{y - x^2}$.

Solution:
(a) For $f(x,y) = x \ln y$, recall that $\ln y$ is defined only for $y > 0$. The domain of f is then the set $D = \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$, that is, the half-plane lying above the x -axis (see Figure 1).

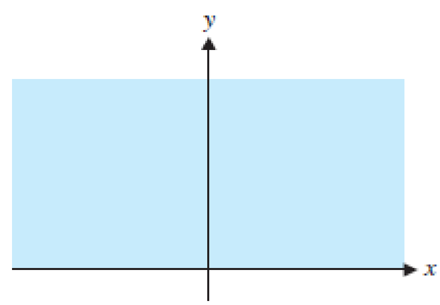


Figure1: the domain of $f(x,y) = x \ln y$

(b) $g(x,y)=\dfrac{2x}{y-x^2}$, note that g is defined unless there is a division by zero, which occurs when $y-x^2=0$. The domain of g is then $D=\{(x,y)\in\mathbb{R}^2 \mid y\neq x^2\}$, which is the entire xy -plane with the parabola $y=x^2$ removed (see Figure 2).

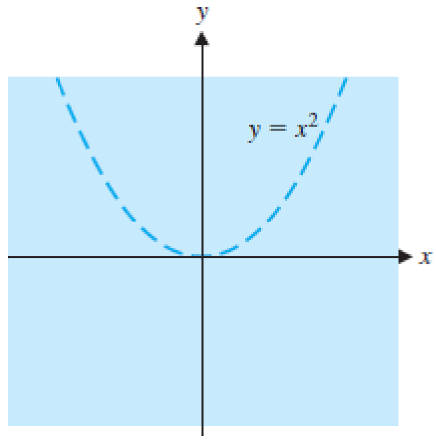


Figure2: the domain of $g(x,y)=\dfrac{2x}{y-x^2}$

Example 2(Finding the Domain of a Function of Three Variables)

Find and describe in graphical terms the domains of

(a) $f(x,y,z)=\dfrac{\cos(x+z)}{xy}$.

(b) $g(x,y,z)=\sqrt{1-x^2-y^2-z^2}$.

Solution

(a) For $f(x,y,z)=\dfrac{\cos(x+z)}{xy}$, there is a division by zero if $xy=0$, which occurs if $x=0$ or $y=0$. The domain is then $D=\{(x,y,z)\in\mathbb{R}^3 \mid x\neq 0 \;\&\; y\neq 0\}$, which is all of three-dimensional space, excluding the yz -plane ($x=0$) and the xz -plane ($y=0$).

(b) Notice that for $g(x,y,z)=\sqrt{1-x^2-y^2-z^2}$ to be defined, you'll need to have $1-x^2-y^2-z^2\geq 0$, or $x^2+y^2+z^2\leq 1$. The domain of g is then the unit sphere of radius 1 centered at the origin and its interior (see Figure 3).

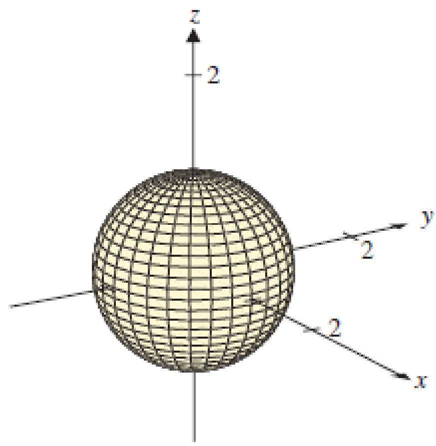


Figure3: the domain of $g(x, y, z) = \sqrt{1 - x^2 - y^2 - z^2}$

Definition 2

The ***graph*** *of the function f (x, y) is the graph of the equation z = f (x, y).*

Example 3 (Graphing Functions of Two Variables)

Graph (a) $f(x, y) = x^2 + y^2$ and (b) $g(x, y) = \sqrt{4 - x^2 + y^2}$.

Solution

(a) For $f(x, y) = x^2 + y^2$, you may recognize the surface $z = x^2 + y^2$ as a circular paraboloid. Notice that the traces in the planes $z = k > 0$ 0 are circles, while the traces in the planes $x = k$ and $y = k$ are parabolas. A graph is shown in Figure 4 .

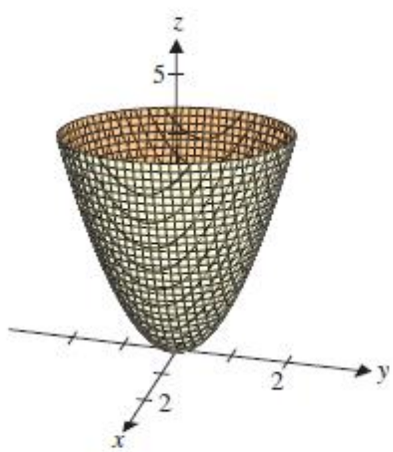


Figure 4: Graph of $z = x^2 + y^2$

(b) For $g(x,y)=\sqrt{4-x^2+y^2}$, note that the surface $z=\sqrt{4-x^2+y^2}$ is the top half of the surface $z^2=4-x^2+y^2$ or $x^2-y^2+z^2=4$. Here, observe that the traces in the planes $x=k$ and $z=k$ are hyperbolas, while the traces in the planes $y=k$ are circles. This gives us a hyperboloid of one sheet, wrapped around the y-axis. The graph of $z=g(x,y)$ is the top half of the hyperboloid, as shown in Figure 5.

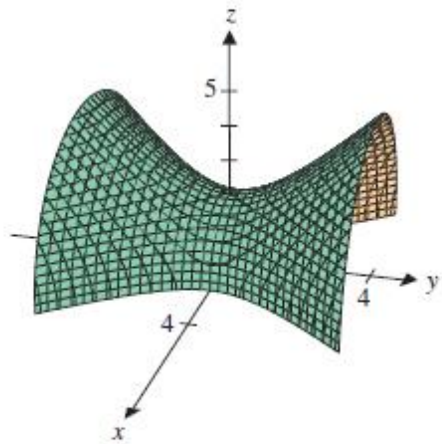


Figure 5: Graph of $z=\sqrt{4-x^2+y^2}$

Definition 3

*A **level curve** of the function $f(x,y)$ is the (two-dimensional) graph of the equation $f(x,y)=c$, for some constant c. (So, the level curve $f(x,y)=c$ is a two-dimensional graph of the trace of the surface $z=f(x,y)$ in the plane $z=c$.)*
*A **contour plot** of $f(x,y)$ is a graph of numerous level curves $f(x,y)=c$, for representative values of c.*

Example 4 (Sketching Contour Plots)

Sketch contour plots for (a) $f(x,y)=-x^2+y$ and (b) $g(x,y)=x^2+y^2$.

Solution

(a) First, note that the level curves of $f(x,y)$ are defined by $-x^2+y=c$, where c is a constant. Solving for y, you can identify the level curves as the parabolas $y=x^2+c$.
A contour plot with $c=-4,-2,0,2$ and 4 is shown in Figure 6.

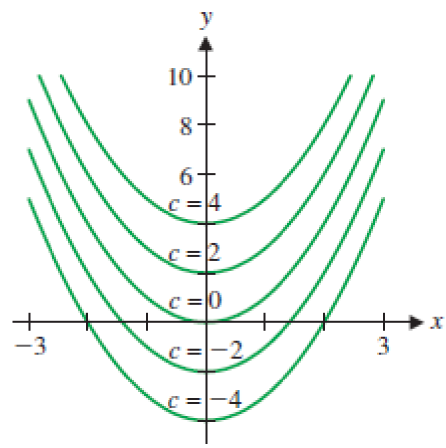


Figure 6: Contour plot $f(x,y) = -x^2 + y$

(b) The level curves for $g(x,y)$ are the circles $x^2 + y^2 = c$. In this case, note that there are level curves *only* for $c \geq 0$. A contour plot with $c = 1, 4, 7$ and 10 is shown in Figure7.

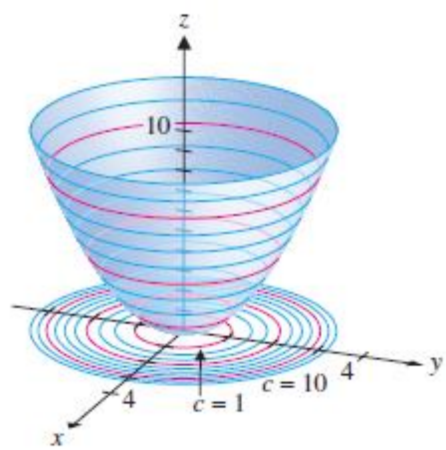


Figure 7: Contour plot $g(x,y) = x^2 + y^2$

12.2 Limits of Functions in Several Variables

Definition 1 (Formal Definition of Limit)
Let f be defined on the interior of a circle centered at the point (a,b), except possibly at (a,b) itself. We say that $\lim_{(x,y)\to(a,b)} f(x,y) = L$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x,y) - L| < \varepsilon$ whenever $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$.
We illustrate the definition in Figure 1.

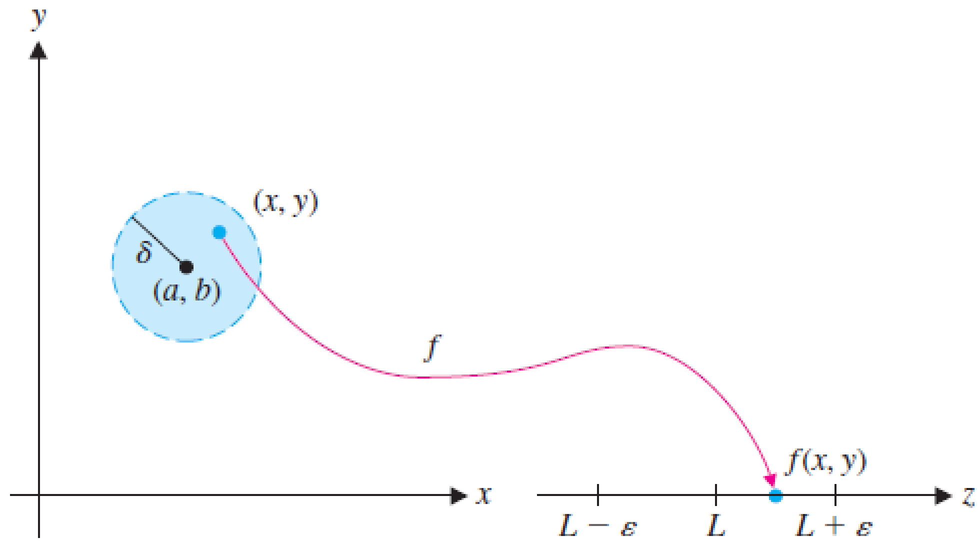


Figure 1: Limit of a Function of Two Variables

Remark 1

The definition of the limit of a function of three variables is completely analogous to the definition for a function of two variables. We say that $\lim_{(x,y,z)\to(a,b,c)} f(x,y,z) = L$, if we can make $f(x,y,z)$ as close as desired to L by making the point (x,y,z) sufficiently close to (a,b,c).

Example 1 (Finding a Simple Limit)

Evaluate $\lim_{(x,y)\to(2,1)} \dfrac{2x^2y+3xy}{5xy^2+3y}$.

Solution

First, note that this is the limit of a rational function (i.e., the quotient of two polynomials). Since the limit in the denominator is

$\lim_{(x,y)\to(2,1)} 5xy^2+3y = 13 \neq 0$, we have $\lim_{(x,y)\to(2,1)} \dfrac{2x^2y+3xy}{5xy^2+3y} = \dfrac{14}{13}$.

Remark 2

- We can show that the limit of any polynomial always exists and is found simply by substitution.
- We can show that the limit of any rational function at a point in its domain always exists and is found simply by substitution.

<u>Theorem 1</u>

If $f(x,y)$ approaches L_1 as (x,y) approaches (a,b) along a path P_1 and $f(x,y)$ approaches $L_2 \neq L_1$ as (x,y) approaches (a,b) along a path P_2, then $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.

Remark 3

Unlike the case for functions of a single variable where we must consider left- and right-hand limits in two dimensions, instead of just two paths approaching a given point,

there are infinitely many (and you obviously can't check each one individually).
In practice, when you suspect that a limit does not exist, you should check the limit along the simplest paths first (Figure 2).

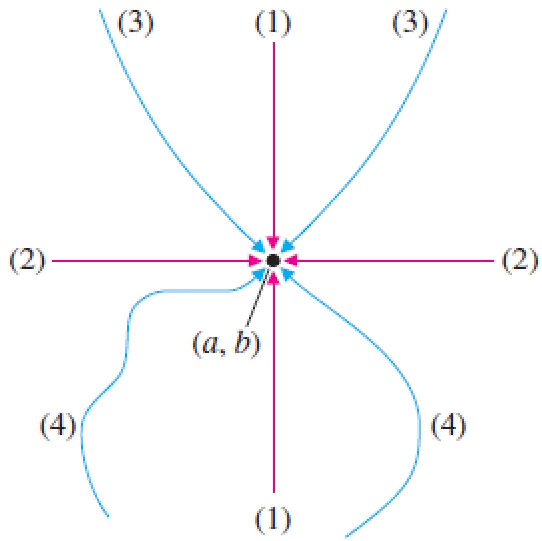


Figure 2: Various paths to (a,b)

Example 2 (A Limit That Does Not Exist)

Evaluate $\lim_{(x,y)\to(1,0)} \frac{y}{x+y-1}$.

Solution

First, we consider the vertical line path along the line $x=1$ and compute the limit as y approaches 0 0. If $(x,y)\to(1,0)$ along the line $x=1$, we have

$$\lim_{y\to 0} \frac{y}{1+y-1} = 1.$$

We next consider the path along the horizontal line $y=0$ and compute the limit as x approaches 1. Here, we have $\lim_{x\to 1} \frac{0}{x+0-1} = 0$

Since the function approaches two different values along two different paths to the point (1, 0), the limit does not exist.

Example 3 (A Limit that is the same along two paths but Does Not Exist)

Evaluate $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$.

Solution

First, we consider the limit along the path $x=0$. We have $\lim_{y\to 0} \frac{0}{0^2+y^2} = 0$.

Similarly, for the path $y = 0$, we have $\lim_{x \to 0} \frac{0}{x^2 + 0^2} = 0$.

Be careful; just because the limits along the first two paths you try are the same does *not* mean that the limit exists. For a limit to exist, the limit must be the same along *all* paths through (0, 0) (not just along two). Here, we may simply need to look at more paths.

Notice that for the path $y = m\,x$ with $m \in \mathbb{R}^*$, we have $\lim_{x \to 0} \frac{mx^2}{x^2 + (mx)^2} = \frac{m}{1+m^2}$.

Since the limit along this path depends of m, the limit does not exist.

Example 4 (A Limit Problem Requiring a More Complicated Choice of Path)

Evaluate $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2 + y^4}$.

Solution

Notice that for the path $x = m\,y^2$ with $m \in \mathbb{R}$ (pass through the origin point $(0,0)$), we have

$$\lim_{y \to 0} \frac{my^4}{(my^2)^2 + y^4} = \frac{m}{m^2 + 1}$$

Since the limit along this path depends of m, the limit does not exist (see Figure 3).

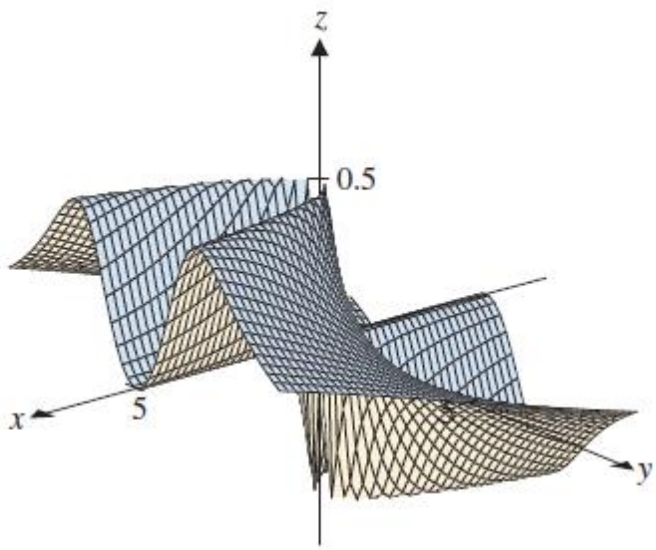


Figure 3: the surface of $z = \frac{xy^2}{x^2 + y^4}$ for $-5 \le x \le 5$ and $-5 \le y \le 5$

<u>Theorem 2</u>

Suppose that $|f(x,y) - L| \le g(x,y)$ *for all* (x,y) *in the interior of some circle centered at* (a,b)*, except possibly at* (a,b) *.*

If $\lim_{(x,y)\to(a,b)} g(x,y) = 0$*, then* $\lim_{(x,y)\to(a,b)} f(x,y) = L$ *.*

Example 5 (Proving That a Limit Exists)

Evaluate $\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2+y^2}$.

Solution

As we did in earlier examples, we start by looking at the limit along several paths through $(0,0)$.

Along the path $x=0$, we have $\lim_{(0,y)\to(0,0)} \frac{0^2 y}{0^2+y^2}=0$.

Similarly, along the path $y=0$, we have $\lim_{(x,0)\to(0,0)} \frac{x^2 .0}{x^2+0^2}=0$.

Further, along the path $y=m\,x$ (with m a real number), we have

$$\lim_{(x,mx)\to(0,0)} \frac{x^2 mx}{x^2+(mx)^2}=\lim_{x\to 0}\frac{mx}{1+m^2}=0.$$

We know that if the limit exists, it must equal 0. After simplifying the expression, there remained an extra power of x in the numerator forcing the limit to 0. To show that the limit equals 0, consider $\left|f(x,y)-0\right|=\left|\frac{x^2 y}{x^2+y^2}\right|$.

Notice that without the y^2 term in the denominator, we could cancel the x^2 terms.

Since $x^2+y^2\ge x^2$, we have that for $x\neq 0$, $\left|f(x,y)-0\right|=\left|\frac{x^2 y}{x^2+y^2}\right|\le\left|\frac{x^2 y}{x^2}\right|\le|y|$.

Since $\lim_{(x,y)\to(0,0)} |y|=0$, Theorem 2 gives us $\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2+y^2}=0$, also.

12.3 Continuity of functions in two or three variables

Definition 1

Suppose that $f(x,y)$ is defined in the interior of a circle centered at the point (a,b).

*We say that f is **continuous** at (a,b) if $\lim_{(x,y)\to(a,b)} f(x,y)=f(a,b)$.*

*If $f(x,y)$ is not continuous at (a,b), then we call (a,b) a **discontinuity** of f .*

*We say that a function $f(x,y)$ is **continuous on a region** R if it is continuous at each point in R.*

Remark 1

- The definition of the continuity of a function of three variables is completely analogous to the definition for a function of two variables:

Suppose that $f(x,y,z)$ is defined in the interior of a sphere centered at (a,b,c). We say that *f* is **continuous** at (a,b,c) if $\lim_{(x,y,z)\to(a,b,c)} f(x,y,z)=f(a,b,c)$

If $f(x,y,z)$ is not continuous at (a,b,c), then we call (a,b,c) a **discontinuity** of f.

- Notice that because we define continuity in terms of limits, we immediately have the following results, which follow directly from the corresponding results for limits. If $f(x,y)$ and $g(x,y)$ are continuous at (a,b), then $f+g$, $f-g$ and $f\cdot g$ are all continuous at (a,b). Further, $\frac{f}{g}$ is continuous at (a,b), if, in addition, $g(a,b)\neq 0$.

Example 1(Determining Where a Function of Two Variables Is Continuous)

Find all points where the given function is continuous:

(a) $f(x,y)=\dfrac{x}{x^2-y}$.

(b) $g(x,y)=\begin{cases}\dfrac{x^4}{x\left(x^2+y^2\right)}, (x,y)\neq(0,0)\\ 0,\quad (x,y)=(0,0)\end{cases}$.

Solution

- For (a), notice that $f(x,y)$ is a quotient of two polynomials (i.e., a rational function) and so, it is continuous at any point where we don't divide by 0. Since division by zero occurs only when $y=x^2$, we have that f is continuous at all points (x,y) with $y\neq x^2$.
- For (b), the function g is also a quotient of polynomials, except at the origin. Notice that there is a division by 0 whenever $x=0$. We must consider the point $(0,0)$ separately, however, since the function is not defined by the rational expression there. We can verify that $\lim\limits_{(x,y)\to(0,0)} g(x,y)=0=g(0,0)$ using the following string of inequalities. Notice that for $(x,y)\neq(0,0)$,

$$|g(x,y)|=\left|\frac{x^4}{x\left(x^2+y^2\right)}\right|\leq\left|\frac{x^4}{x\left(x^2\right)}\right|=|x|$$

and $|x|\to 0$ as $(x,y)\to(0,0)$. We deduce that $\lim\limits_{(x,y)\to(0,0)} g(x,y)=0=g(0,0)$, so that g is continuous at $(0,0)$. Putting this all together, we get that g is continuous at the origin and also at all points (x,y) with $x\neq 0$.

<u>**Theorem 1**</u>

Suppose that $f(x,y)$ is continuous at (a,b) and $g(x)$ is continuous at the point $f(a,b)$. Then $h(x,y)=g\circ f(x,y)=g(f(x,y))$ is continuous at (a,b).

Example 2 (Determining Where a Composition of Functions Is Continuous)

Determine where $f(x,y)=e^{x^2y}$ is continuous?

Solution

Notice that $f(x,y) = g(h(x,y))$, where $g(t) = e^t$ and $h(x,y) = x^2 y$. Since g is continuous for all values of t and h is a polynomial in x and y (and hence continuous on $\mathbb{R}^2$), it follows from Theorem 1 that f is continuous on $\mathbb{R}^2$.

Example 3

Determine where $h(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$ is continuous ?

Solution

- The function $f(x,y) = \frac{y}{x}$ is a rational function and therefore continuous except on the line $x = 0$.

- The function $g(t) = \tan^{-1} t$ is continuous everywhere.

It follows from Theorem 1 that h is continuous on $\mathbb{R}^2 \setminus \{x = 0\}$ (see Figure1).

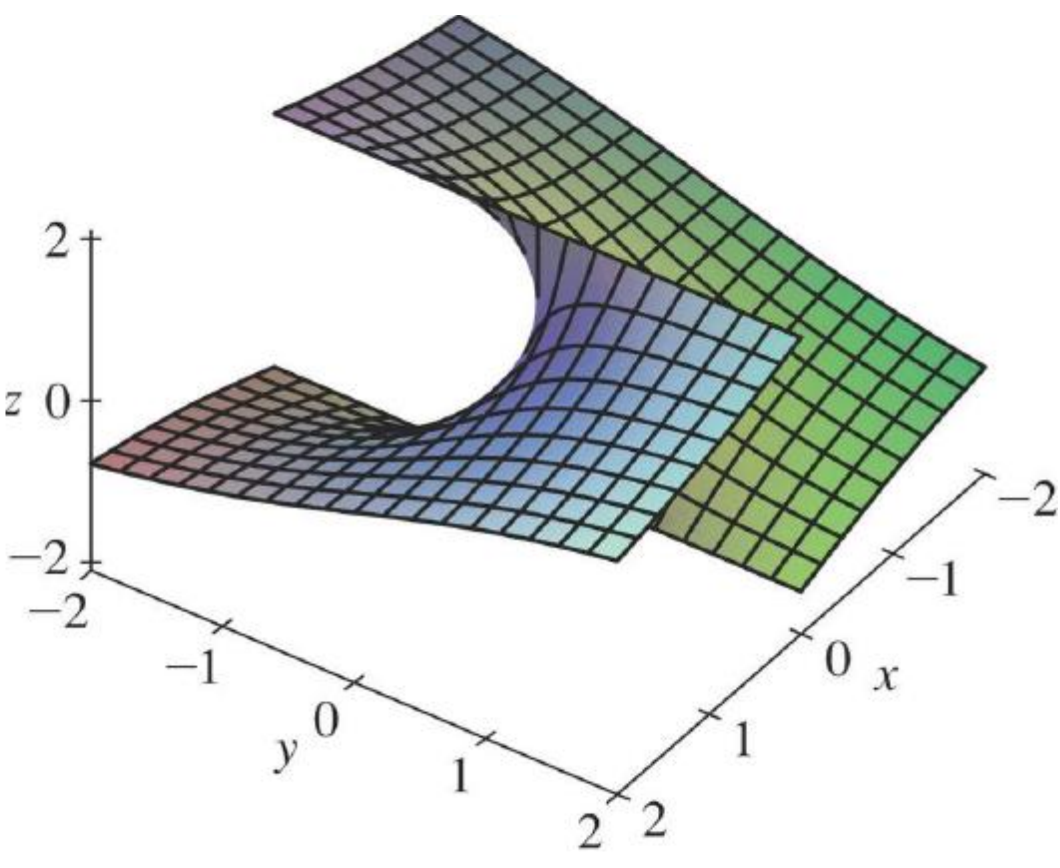


Figure1: the figure shows the break in the graph of $h(x,y)=\tan^{-1}\left(\frac{y}{x}\right)$ above the y-axis

Example 4

Determine where $f(x,y)=\begin{cases}\frac{3x^2y}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$ is continuous?

Solution

We know f is continuous for $(x,y)\neq(0,0)$. Since it is equal to a rational function there.

Also we have $\lim_{(x,y)\to(0,0)}\frac{3x^2y}{x^2+y^2}=0=f(0,0)$. Thus f is continuous at $(0,0)$.

So f is continuous on $\mathbb{R}^2$ (see Figure 2).

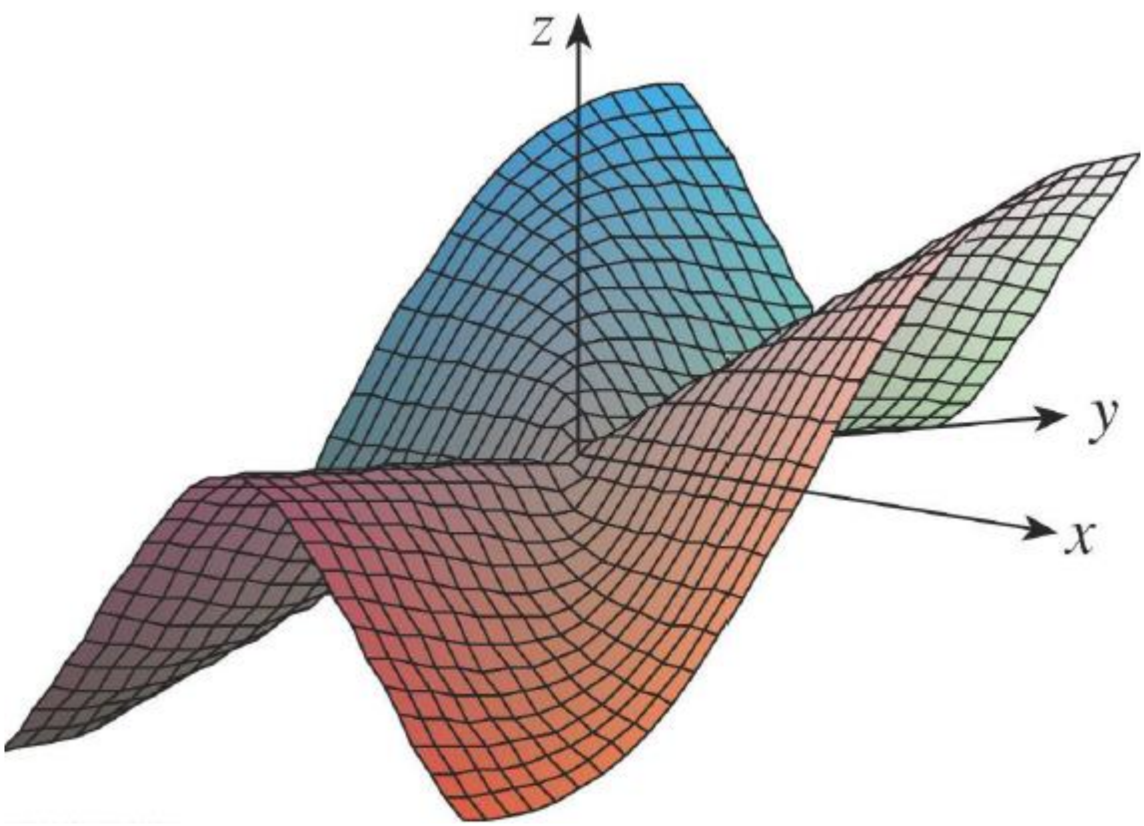


Figure2: Graph of $z = f(x,y) = \begin{cases} \dfrac{3x^2 y}{x^2+y^2}, (x,y) \neq (0,0) \\ 0, \quad (x,y) = (0,0) \end{cases}$.

Example 5

Determine where $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, (x,y) \neq (0,0) \\ 0, \quad (x,y) = (0,0) \end{cases}$ is continuous?

Solution

We know f is continuous for $(x,y) \neq (0,0)$. Since it is equal to a rational function there. Also we have $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$ does not exist. Thus f is not continuous at $(0,0)$. So f is continuous on $\mathbb{R}^2 \setminus \{(0,0)\}$ (see Figure3).

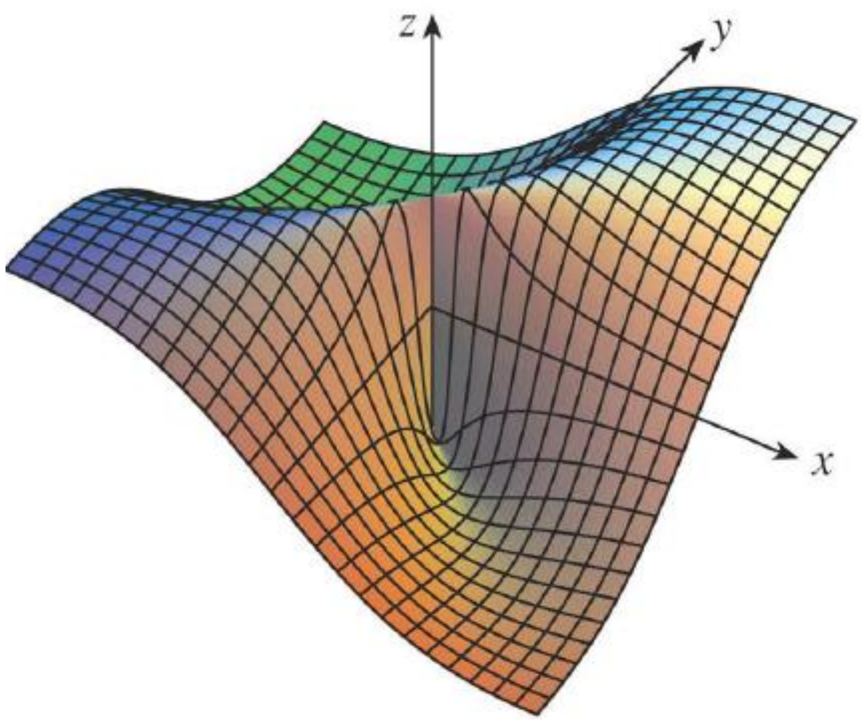


Figure3: Graph of $z = f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, (x,y) \neq (0,0) \\ 0, \quad (x,y) = (0,0) \end{cases}$.

Example 6 (Continuity for a Function of Three Variables)

Find all points where $f(x,y,z) = \ln\left(9 - x^2 - y^2 - z^2\right)$ is continuous.

Solution

Notice that $f(x,y,z)$ is defined only for $9 - x^2 - y^2 - z^2 > 0$. On this domain, f is a composition of continuous functions, which is also continuous. So, f is continuous for $x^2 + y^2 + z^2 < 9$, which you should recognize as the interior of the sphere of radius 3 centered at (0, 0, 0).

12.4 First-order partial derivatives

In this section, we generalize the notion of derivative to functions of more than one variable.

First, recall that for a function f of a single variable, we define the derivative function as $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$, for any values of *x* for which the limit exists.

At any particular value $x=a$, we interpret $f'(a)$ as the instantaneous rate of change of the function with respect to x at that point.

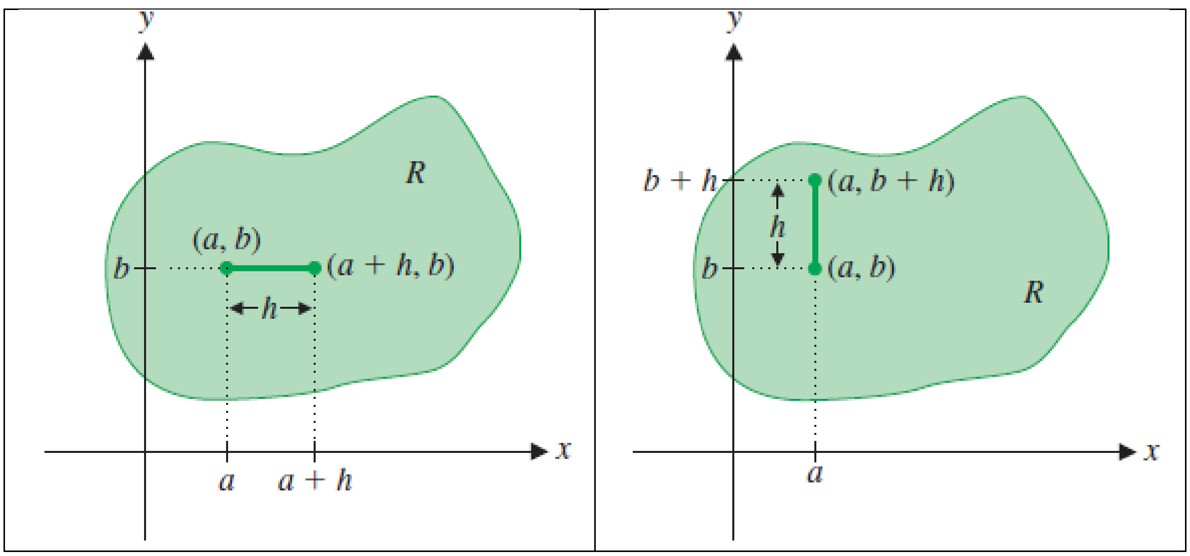
Definition 1

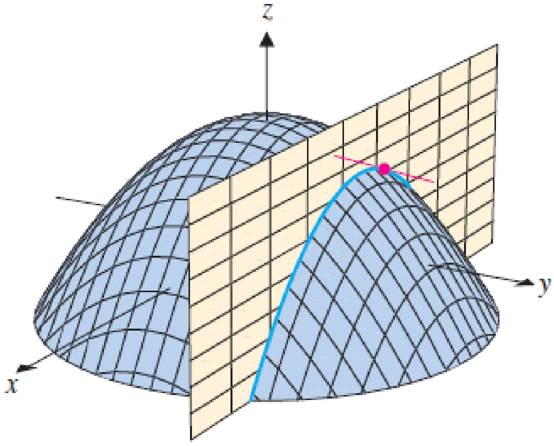
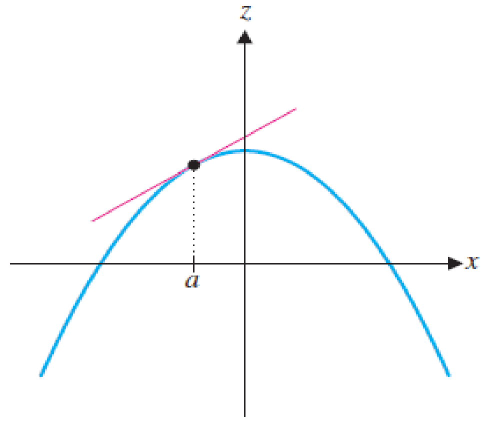
The ***partial derivative of*** $f(x,y)$ ***with respect to*** x *, written* $\frac{\partial f}{\partial x}$ *, is defined by*

$\frac{\partial f}{\partial x}(x,y)=\lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$*, for any values of* x *and* y *for which the limit exists.*

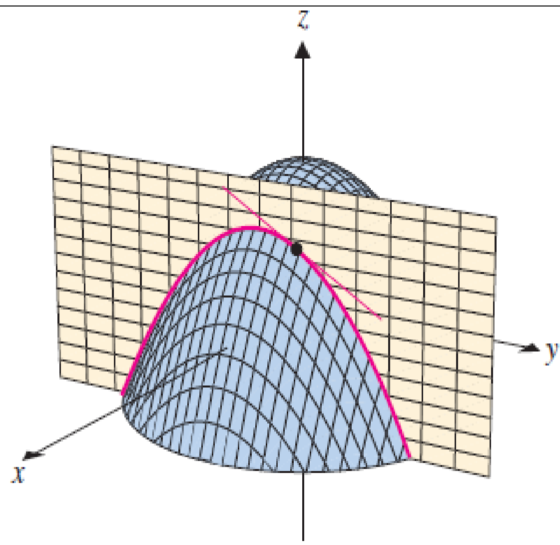
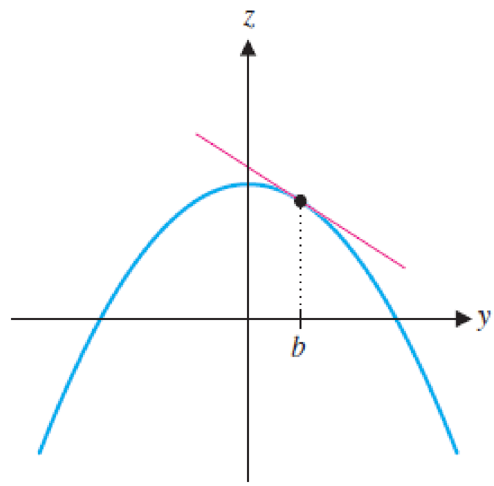
The ***partial derivative of*** $f(x,y)$ ***with respect to*** y *, written* $\frac{\partial f}{\partial y}$*, is defined by,*

$\frac{\partial f}{\partial y}(x,y)=\lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}$*, for any values of x and y for which the limit exists.*



 Intersection of the surface $z = f(x, y)$ with the plane $y = b$.	 The curve $z = f(x, b)$.

- $\dfrac{\partial f}{\partial x}(a,b)$ gives the slope of the tangent line to the curve at $x = a$.

 Intersection of the surface $z = f(x, y)$ with the plane $x = a$.	 The curve $z = f(a, y)$.

- $\dfrac{\partial f}{\partial y}(a,b)$ gives the slope of the tangent line to the curve at $y = b$.

Remark 1

- To compute the partial derivative $\frac{\partial f}{\partial x}$, you simply take an ordinary derivative with respect to x , while treating y as a constant. Similarly, you compute $\frac{\partial f}{\partial y}$ by taking an ordinary derivative with respect to y , while treating x as a constant.
- For $z = f(x,y)$, we write $\frac{\partial f}{\partial x}(x,y) = f_x(x,y) = \frac{\partial z}{\partial x}(x,y) = \frac{\partial}{\partial x}\left[f(x,y)\right]$.
- The expression $\frac{\partial}{\partial x}$ is a **partial differential operator.** It tells you to take the partial derivative (with respect to x) of whatever expression follows it. Similarly, we have

$$\frac{\partial f}{\partial y}(x,y) = f_y(x,y) = \frac{\partial z}{\partial y}(x,y) = \frac{\partial}{\partial y}\left[f(x,y)\right].$$

Example 1 (Computing Partial Derivatives)

For $f(x,y) = 3x^2 + x^3y + 4y^2$, compute $\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y), f_x(1,0)$ and $f_y(2,-1)$.

Solution

Compute $\frac{\partial f}{\partial x}$ by treating y as a constant. We have

$$\frac{\partial f}{\partial x}(x,y) = \frac{\partial}{\partial x}\left[3x^2 + x^3y + 4y^2\right] = 6x + 3x^2y.$$

The partial derivative of $4y^2$ with respect to x is 0 , since $4y^2$ is treated as if it were a constant when differentiating with respect to x . Next, we compute $\frac{\partial f}{\partial y}$ by treating x as a constant. We have

$$\frac{\partial f}{\partial y}(x,y) = \frac{\partial}{\partial y}\left[3x^2 + x^3y + 4y^2\right] = x^3 + 8y.$$

Substituting values for x and y , we get $f_x(1,0) = \frac{\partial f}{\partial x}(1,0) = 6$ and $f_y(2,-1) = \frac{\partial f}{\partial y}(2,-1) = 0.$

Remark 2

Since we are holding one of the variables fixed when we compute a partial derivative, we have the product rules: $\frac{\partial}{\partial x}(uv) = \frac{\partial u}{\partial x}v + u\frac{\partial v}{\partial x}$ and $\frac{\partial}{\partial y}(uv) = \frac{\partial u}{\partial y}v + u\frac{\partial v}{\partial y}$

and the quotient rule: $\frac{\partial}{\partial x}\left(\frac{u}{v}\right)=\frac{\frac{\partial u}{\partial x}v-u\frac{\partial v}{\partial x}}{v^2}$,

with a corresponding quotient rule holding for $\frac{\partial}{\partial y}\left(\frac{u}{v}\right)=\frac{\frac{\partial u}{\partial y}v-u\frac{\partial v}{\partial y}}{v^2}$.

Example 2 (Computing Partial Derivatives)

For $f(x,y)=e^{xy}+\frac{x}{y}$, compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

Solution

For $y\neq 0$, we have $\frac{\partial f}{\partial x}(x,y)=\frac{\partial}{\partial x}\left[e^{xy}+\frac{x}{y}\right]=y\,e^{xy}+\frac{1}{y}$. Also,

$\frac{\partial f}{\partial y}(x,y)=\frac{\partial}{\partial y}\left[e^{xy}+\frac{x}{y}\right]=x\,e^{xy}-\frac{x}{y^2}$.

Example 3 (Computing Partial Derivatives)

For $f(x,y,z)=\sin(x^2y^3z)+xy\ln z$, compute $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$.

Solution

For $z>0$, we have

$\frac{\partial f}{\partial x}(x,y,z)=\frac{\partial}{\partial x}\left[\sin\left(x^2y^3z\right)+xy\ln z\right]=2xy^3z\cos(x^2y^3z)+y\ln z$.

Also,

$\frac{\partial f}{\partial y}(x,y,z)=\frac{\partial}{\partial y}\left[\sin\left(x^2y^3z\right)+xy\ln z\right]=3x^2y^2z\cos(x^2y^3z)+x\ln z$.

And, $\frac{\partial f}{\partial z}(x,y,z)=\frac{\partial}{\partial z}\left[\sin\left(x^2y^3z\right)+xy\ln z\right]=x^2y^3\cos(x^2y^3z)+\frac{xy}{z}$.

12.5 Higher-order partial derivatives

Notice that the partial derivatives found in the preceding examples are themselves functions of two variables. We have seen that second- and higher-order derivatives of functions of a single variable provide much significant information. Not surprisingly, **higher-order partial derivatives** are also very important in applications.

For functions of two variables, there are four different second-order partial derivatives. The partial derivative with respect to x of $\frac{\partial f}{\partial x}$ is $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$, usually abbreviated as $\frac{\partial^2 f}{\partial x^2}$ or f_{xx} . Similarly, taking two successive partial derivatives with respect to y gives us $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial y^2}=f_{yy}$.

For **mixed second-order partial derivatives,** one derivative is taken with respect to each variable. If the first partial derivative is taken with respect to x, we have $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$, abbreviated as $\frac{\partial^2 f}{\partial y\,\partial x}$, or $\left(f_x\right)_y = f_{xy}$. If the first partial derivative is taken with respect to y, we have $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$, abbreviated as $\frac{\partial^2 f}{\partial x\,\partial y}$, or $\left(f_y\right)_x = f_{yx}$.

Example 1 (Computing Second-Order Partial Derivatives)

Find all second-order partial derivatives of $f(x, y) = x^2 y - y^3 + \ln x$.

Solution

We start by computing the first-order partial derivatives: For $x > 0$,

$\frac{\partial f}{\partial x}(x, y) = 2xy + \frac{1}{x}$ and $\frac{\partial f}{\partial y}(x, y) = x^2 - 3y^2$. We then have

$$\frac{\partial^2 f}{\partial x^2}(x, y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(2xy + \frac{1}{x}\right) = 2y - \frac{1}{x^2},$$

$$\frac{\partial^2 f}{\partial y\,\partial x}(x, y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(2xy + \frac{1}{x}\right) = 2x,$$

$$\frac{\partial^2 f}{\partial x\,\partial y}(x, y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(x^2 - 3y^2\right) = 2x,$$

and finally, $\frac{\partial^2 f}{\partial y^2}(x, y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(x^2 - 3y^2\right) = -6y$.

Remark 1

Notice in example 1 that $\frac{\partial^2 f}{\partial y\,\partial x}(x, y) = \frac{\partial^2 f}{\partial x\,\partial y}(x, y)$. It turns out that this is true for most, but *not all*, of the functions that you will encounter.

Theorem 1

If $f_{xy}(x, y)$ and $f_{yx}(x, y)$ are continuous on an open set containing (a,b), then $f_{xy}(a,b) = f_{yx}(a,b)$.

Example 2 (Computing Higher-Order Partial Derivatives)

For $f(x, y) = \cos(xy) - x^3 + y^4$, compute f_{xyy} and f_{xyyy}.

Solution

We have $f_x = \frac{\partial}{\partial x}\left(\cos(xy) - x^3 + y^4\right) = -y\sin(xy) - 3x^2$.

Differentiating f_x with respect to y gives us

$$f_{xy} = \frac{\partial}{\partial y}\left(-y\sin(xy) - 3x^2\right) = -\sin(xy) - xy\cos(xy)$$ and

$$\begin{aligned} f_{xyy} &= \frac{\partial}{\partial y}\left(-\sin(xy) - xy\cos(xy)\right) \\ &= -x\cos(xy) - x\cos(xy) + x^2 y\sin(xy) \\ &= -2x\cos(xy) + x^2 y\sin(xy). \end{aligned}$$

Finally, we have

$$\begin{aligned} f_{xyyy} &= \frac{\partial}{\partial y}\left(-2x\cos(xy) + x^2 y\sin(xy)\right) \\ &= 2x^2\sin(xy) + x^2\sin(xy) + x^3 y\cos(xy) \\ &= 3x^2\sin(xy) + x^3 y\cos(xy). \end{aligned}$$

Example 3 (Partial Derivatives of Functions of Three Variables)

For $f(x,y,z) = \sqrt{xy^3 z} + 4x^2 y$, defined for $x, y, z \geq 0$, compute f_x , f_{xy} and f_{xyz} .

Solution

To keep x , y and z as separate as possible, we first rewrite f as

$$f(x,y,z) = x^{1/2} y^{3/2} z^{1/2} + 4x^2 y .$$

To compute the partial derivative with respect to x , we treat y and z as constants and obtain $f_x = \frac{\partial}{\partial x}\left[x^{1/2} y^{3/2} z^{1/2} + 4x^2 y\right] = \left(\frac{1}{2}x^{-1/2}\right) y^{3/2} z^{1/2} + 8xy .$

Next, treating x and z as constants, we get

$$f_{xy} = \frac{\partial}{\partial y}\left[\frac{1}{2}x^{-1/2} y^{3/2} z^{1/2} + 8xy\right] = \left(\frac{1}{2}x^{-1/2}\right)\left(\frac{3}{2}y^{1/2}\right) z^{1/2} + 8x .$$

Finally, treating x and y as constants, we get

$$\begin{aligned} f_{xyz} &= \frac{\partial}{\partial z}\left[\frac{3}{4}x^{-1/2} y^{1/2} z^{1/2} + 8x\right] = \left(\frac{1}{2}x^{-1/2}\right)\left(\frac{3}{2}y^{1/2}\right)\left(\frac{1}{2}z^{-1/2}\right) \\ &= \frac{3}{8}x^{-1/2} y^{1/2} z^{-1/2}. \end{aligned}$$

Notice that this derivative is defined for $x, z > 0$ and $y \geq 0$. Further, you can show that all first-, second- and third-order partial derivatives are continuous for $x, y, z > 0$, so that the order in which we take the partial derivatives is irrelevant in this case.

12.6 Tangent planes and Linear approximations

Recall that the tangent line to the curve $y = f(x)$ at $x = a$ stays close to the curve near the point of tangency. This enables us to use the tangent line to approximate values of the function close to the point of tangency (see Figure 1).

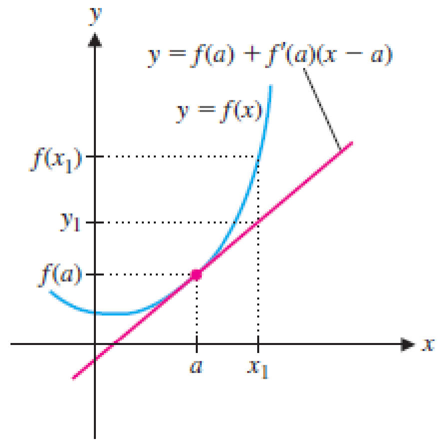


Figure1: Linear approximation.

The equation of the tangent line is given by: $y = f(a) + f'(a)(x - a)$. We called this the *linear approximation* to $f(x)$ at $x = a$.
In much the same way, we can approximate the value of a function of two variables near a given point using the tangent *plane* to the surface at that point. For instance, the graph of $z = 6 - x^2 - y^2$ and its tangent plane at the point $(1,2,1)$ are shown in Figure 2.

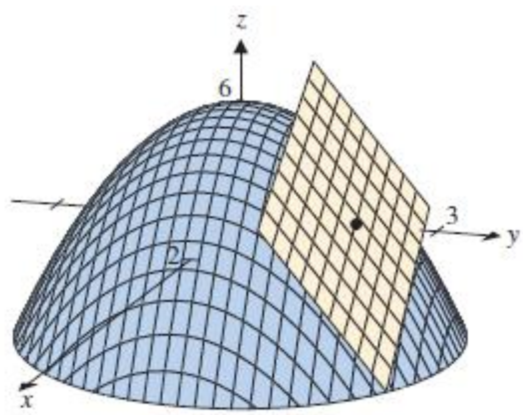


Figure2: $z = 6 - x^2 - y^2$ and the tangent plane at $(1,2,1)$.

Notice that near the point $(1,2,1)$, the surface and the tangent plane are very close together.

Theorem1

Suppose that $f(x,y)$ has continuous first partial derivatives at (a,b). A normal vector to the tangent plane to $z = f(x,y)$ at (a,b) is then $\left(f_x(a,b), f_y(a,b), -1\right)$.
Further, an equation of the tangent plane is given by
$z - f(a,b) = f_x(a,b)(x-a) + f_y(a,b)(y-b)$ ***or***
$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$***.***

Remark1

- A **vector normal** to the plane is then given by the cross product:
$$(0,\ 1, f_y(a,b)) \quad \times \quad (0, f_x(a,b), 1) = (f_x(a,b), f_y(a,b), -1) \ .$$
- The line orthogonal to the tangent plane and passing through the point $(a, b, f(a,b))$ is given by $\begin{cases} x = a + t\, f_x(a,b) \\ y = b + t\, f_y(a,b) \\ z = f(a,b) - t \end{cases}$.

This line is called the **normal line** to the surface at the point $(a, b, f(a,b))$.

Example1 (Finding Equations of the Tangent Plane and the Normal Line)

Find equations of the tangent plane and the normal line to $z = 6 - x^2 - y^2$ at the point $(1,2,1)$.

Solution

For $f(x, y) = 6 - x^2 - y^2$, we have $f_x = -2x$ and $f_y = -2y$. This gives us $f_x(1,2) = -2$ and $f_y(1,2) = -4$. So a normal vector is then $(-2,-4,-1)$.

An equation of the tangent plane is: $z = 1 - 2(x-1) - 4(y-2)$.

Equations of the normal line are $\begin{cases} x = 1 + 2t \\ y = 2 - 4t\,, \quad t \in \mathbb{R}. \\ z = 1 - t \end{cases}$

A sketch of the surface, the tangent plane and the normal line is shown in Figure 3.

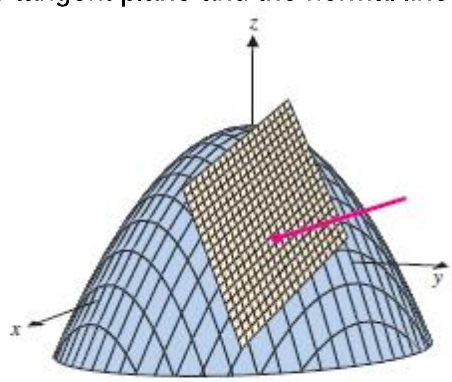


Figure3: Surface, tangent plane and normal line at the point $(1,2,1)$.

Example2 (Finding Equations of the Tangent Plane and the Normal Line)

Find equations of the tangent plane and the normal line to $z = x^3 + y^3 + \dfrac{x^2}{y}$ at the point $(2,1,13)$.

Solution

Here, $f_x = 3x^2 + \frac{2x}{y}$ and $f_y = 3y^2 - \frac{x^2}{y^2}$, so that $f_x(2,1) = 12 + 4 = 16$ and $f_y(2,1) = 3 - 4 = -1$. So a normal vector is then $(16,-1,-1)$.

An equation of the tangent plane is: $z = 13 + 16(x-2) - (y-1)$.

Equations of the normal line are $\begin{cases} x = 2+16t \\ y = 1-t \quad , \quad t \in \mathbb{R}. \\ z = 13-t \end{cases}$

A sketch of the surface, the tangent plane and the normal line is shown in Figure 4.

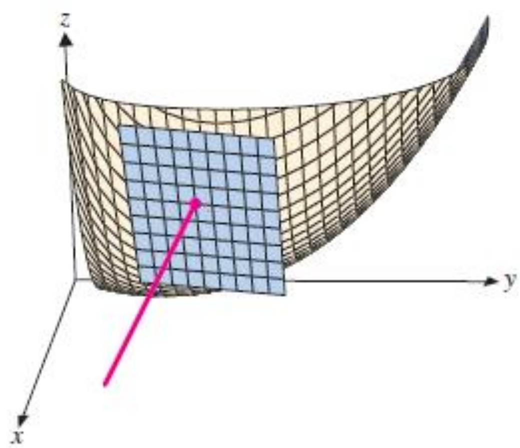


Figure4: Surface, tangent plane and normal line at the point $(2,1,13)$.

12.7 Increments and Differentials

First, we remind you of the notation that we used for functions of a single variable. We defined the **increment** Δy of the function $f(x)$ at $x=a$ to be $\Delta y = f(a+\Delta x) - f(a)$. Referring to Figure 1, notice that for Δx small, $\Delta y \approx dy = f'(a)\Delta x$, where we referred to dy as the **differential** of y.

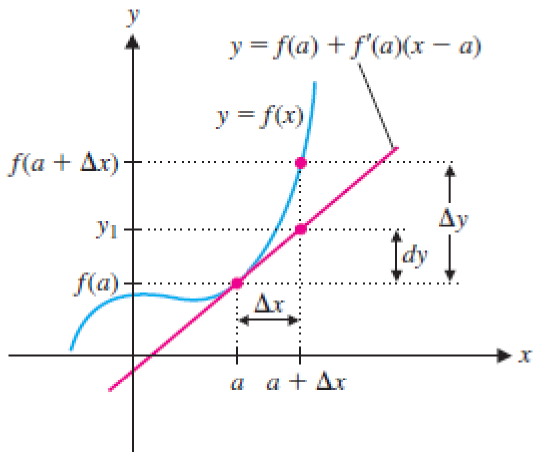


Figure 1: Increments and differentials for a function of one variable.

For $z = f(x,y)$, we define the **increment** of f at (a,b) to be

$$\Delta z = f(a+\Delta x, b+\Delta y) - f(a,b).$$

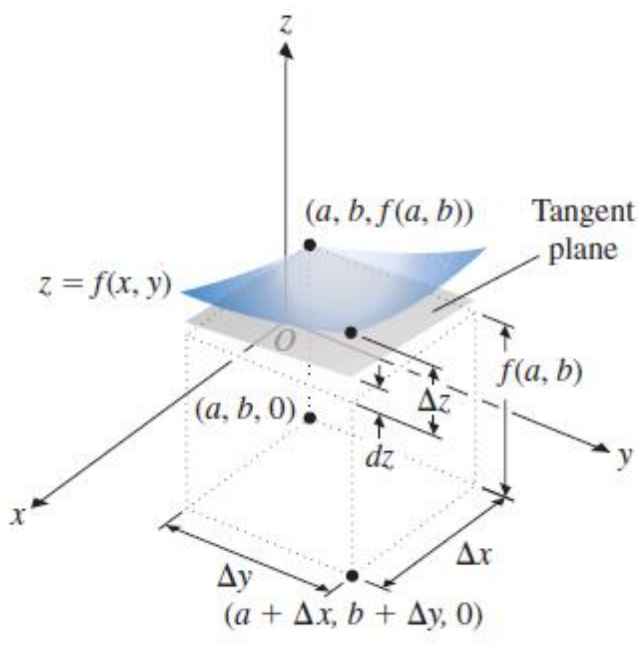


Figure 2: Linear approximation.

Notice that as long as f is continuous in some open region containing (a,b) and f has first partial derivatives on that region, we can write:

$$\Delta z = f(a+\Delta x, b+\Delta y) - f(a,b)$$

$$= \left[f(a+\Delta x, b+\Delta y) - f(a, b+\Delta y)\right] + \left[f(a, b+\Delta y) - f(a,b)\right]$$

Adding and subtracting $f(a, b+\Delta y)$.

$$= f_x(u, b+\Delta y)\left[(a+\Delta x) - a\right] + f_y(a,v)\left[(b+\Delta y) - b\right]$$

Applying the Mean Value Theorem to both terms.

$$= f_x(u, b+\Delta y)\Delta x + f_y(a,v)\Delta y,$$

by the Mean Value Theorem. Here, u is some value between a and $a+\Delta x$, and v is some value between b and $b+\Delta y$ (see Figure 3). This gives us

$$\Delta z = f_x(u, b+\Delta y)\Delta x + f_y(a,v)\Delta y,$$

$$= \left\{f_x(a,b) + \left[f_x(u, b+\Delta y) - f_x(a,b)\right]\right\}\Delta x + \left\{f_y(a,b) + \left[f_y(a,v) - f_y(a,b)\right]\right\}\Delta y$$

which we rewrite as $\Delta z = f_x(a,b)\Delta x + f_y(a,b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$, where $\varepsilon_1 = \left[f_x(u, b+\Delta y) - f_x(a,b)\right]$ and $\varepsilon_2 = \left[f_y(a,v) - f_y(a,b)\right]$.

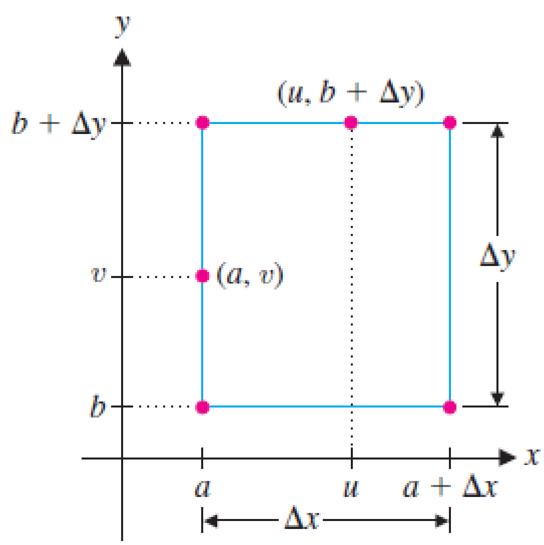


Figure 3: Intermediate points from the Mean Value Theorem.

We have now established the following result.

Theorem1

Suppose that $z = f(x,y)$ ***is defined on the rectangular region*** $R = \left\{(x,y) \in \mathbb{R}^2 \mid x_0 < x < x_1 \,\&\, y_0 < y < y_1\right\}$ ***and*** f_x ***and*** f_y ***are defined on*** R ***and are continuous at*** $(a,b) \in R$ ***. Then for*** $(a+\Delta x, b+\Delta y) \in R$ ***,*** $\Delta z = f_x(a,b)\Delta x + f_y(a,b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$ ***where*** ε_1 ***and*** ε_2 ***are functions of*** Δx ***and*** Δy ***that both tend to zero, as*** $(\Delta x, \Delta y) \to (0,0)$***.***

Example 1 (Computing the Increment Δz)

For $z = f(x, y) = x^2 - 5xy$, find Δz .

Solution

We have

$$\Delta z = f\left(x + \Delta x, y + \Delta y\right) - f\left(x, y\right).$$
$$= \left(x + \Delta x\right)^2 - 5\left(x + \Delta x\right)\left(y + \Delta y\right) - \left[x^2 - 5xy\right].$$
$$= x^2 + 2x\ \Delta x + \left(\Delta x\right)^2 - 5\left(xy + x\ \Delta y + y\ \Delta x + \Delta x\ \Delta y\right) - x^2 + 5xy$$
$$= \left(2x - 5\right)\Delta x + \left(-5x\right)\Delta y + \left(\Delta x\right)\Delta x + \left(-5\Delta x\right)\Delta y.$$
$$= f_x(x, y)\ \Delta x + f_y(x, y)\ \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y,$$

where $\varepsilon_1 = \Delta x$ and $\varepsilon_2 = -5\Delta x$ both tend to zero, as $\left(\Delta x, \Delta y\right) \to \left(0, 0\right)$.

Example 2

Let $z = f(x, y) = 3x^2 - xy$.

(a) If Δx and Δy are increments of x and y , find Δz .

(b) Use Δz to calculate the change in $f(x, y)$ if $\left(x, y\right)$ changes from $\left(1, 2\right)$ to $\left(1.01, 1.98\right)$.

Solution

(a) We have

$$\Delta z = f\left(x + \Delta x, y + \Delta y\right) - f\left(x, y\right).$$
$$= 3\left(x + \Delta x\right)^2 - \left(x + \Delta x\right)\left(y + \Delta y\right) - \left[3x^2 - xy\right].$$
$$= 3x^2 + 6x\ \Delta x + 3\left(\Delta x\right)^2 - \left(xy + x\ \Delta y + y\ \Delta x + \Delta x\ \Delta y\right) - 3x^2 + xy$$
$$= \left(6x - y\right)\Delta x + \left(-x\right)\Delta y + \left(3\Delta x\right)\Delta x + \left(-\Delta x\right)\Delta y.$$
$$= f_x(x, y)\ \Delta x + f_y(x, y)\ \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y,$$

where $\varepsilon_1 = 3\Delta x$ and $\varepsilon_2 = -\Delta x$ both tend to zero, as $\left(\Delta x, \Delta y\right) \to \left(0, 0\right)$.

(b) If $\left(x, y\right)$ changes from $\left(1, 2\right)$ to $\left(1.01, 1.98\right)$, substituting $x = 1, y = 2, \Delta x = 0.01,$ and $\Delta y = -0.02$ into the formula for Δz gives us

$$\Delta z = \left[6(1) - 2\right](0.01) - (1)(-0.02) + 3(0.01)^2 - (0.01)(-0.02) = 0.0605.$$

Remark1

If we increment x by the amount $dx = \Delta x$ and increment y by $dy = \Delta y$, then we define the **total differential** of z to be $dz = f_x(x, y)dx + f_y(x, y)dy$.

Definition1

*Let $z = f(x, y)$. We say that f is **differentiable** at $\left(a, b\right)$ if we can write $\Delta z = f_x(a, b)\ \Delta x + f_y(a, b)\ \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$, where ε_1 and ε_2 are both functions of Δx and Δy and $\varepsilon_1, \varepsilon_2 \to 0$, as $\left(\Delta x, \Delta y\right) \to \left(0, 0\right)$. We say that f is differentiable on a region $R \subseteq \mathbb{R}^2$ whenever f is differentiable at every point in R .*

Definition2

*The **linear approximation** to* $f(x,y,z)$ *at the point* (a,b,c) *is given by*

$L(x,y,z)=f(a,b,c)+f_x(a,b,c)(x-a)+f_y(a,b,c)(y-b)+f_z(a,b,c)(z-c).$

Example 3

The dimensions of a closed rectangular box are measured as 3 feet, 4 feet, and 5 feet, with a possible error of $\pm\frac{1}{16}$ inch in each measurement. Use differentials to approximate the maximum error in the calculated value of

(a) The surface area.

(b) The volume.

Solution

(a) The surface area is $S=2(xy+yz+xz)$. So

$$dS=2(y+z)dx+2(x+z)dy+2(x+y)dz.$$

As $dx=dy=dz=\pm\frac{1}{16}$ inch $=\pm\frac{1}{192}$ feet, we get $dS=(18+16+14)\left(\frac{\pm 1}{192}\right)=\pm\frac{1}{4}$ feet^2.

(b) The volume is $V=x\ y\ z$. So

$$\begin{aligned} dV &= yz\ dx+xz\ dy+xy\ dz \\ &= (20+15+12)\left(\frac{\pm 1}{192}\right)=\pm\frac{47}{192}\text{ feet}^3. \end{aligned}$$

12.8 Chain Rule and Implicit Differentiation

The general form of the chain rule says that for differentiable functions f and g ,

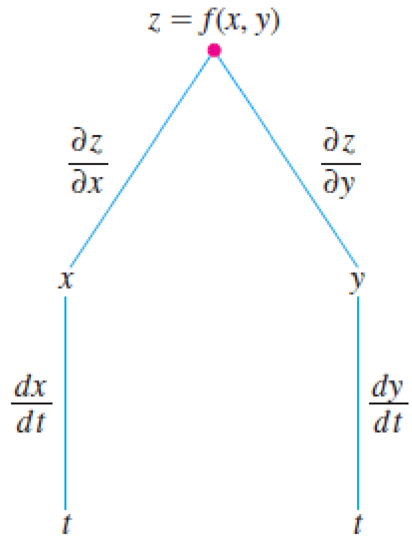
$$\frac{d}{dx}\left[f\left(g(x)\right)\right]=f'\left(g(x)\right)g'(x).$$

We now extend the chain rule to functions of several variables.

Theorem1 (Chain Rule)

If $z=f\left(x(t),y(t)\right)$**, where** $x(t)$ **and** $y(t)$ **are differentiable and** $f(x,y)$ **is a differentiable function of** x **and** y **, then**

$$\frac{dz}{dt}=\frac{d}{dt}\left[f\left(x(t),y(t)\right)\right]=\frac{\partial f}{\partial x}\left(x(t),y(t)\right)\frac{dx}{dt}+\frac{\partial f}{\partial y}\left(x(t),y(t)\right)\frac{dy}{dt}\ .$$



$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}$$

Example1 (Using the Chain Rule)

For $z=f(x,y)=x^2e^y$, $x(t)=t^2-1$ and $y(t)=\sin t$, find the derivative of $g(t)=f\left(x(t),y(t)\right)$.

Solution

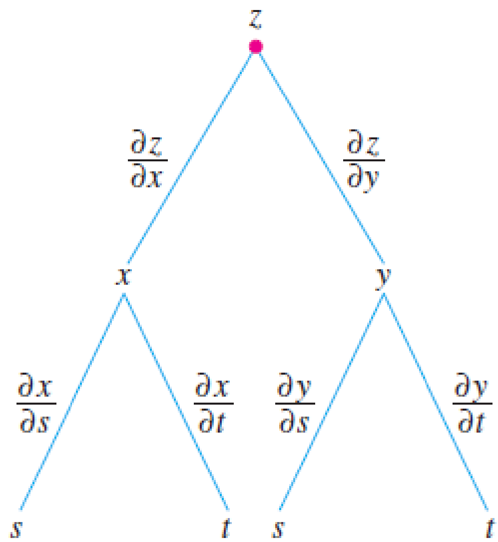
We first compute the derivatives $\frac{\partial z}{\partial x}=2xe^y$, $\frac{\partial z}{\partial y}=x^2e^y$, $x'(t)=2t$ and $y'(t)=\cos t$.

The chain rule (Theorem1) then gives us

$$g'(t)=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}=2xe^y\left(2t\right)+x^2e^y\left(\cos t\right)$$
$$=4t\left(t^2-1\right)e^{\sin t}+\left(\cos t\right)\left(t^2-1\right)^2e^{\sin t}\ .$$

Theorem2 (Chain Rule)

Suppose that $z=f(x,y)$, where f is a differentiable function of x and y and where $x=x(s,t)$ and $y=(s,t)$ both have first-order partial derivatives. Then we have the chain rules: $\frac{\partial z}{\partial s}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$ and $\frac{\partial z}{\partial t}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$.



Example 2 (Using the Chain Rule)

Suppose that $f(x,y)=e^{xy}$, $x(u,v)=3u\sin v$ and $y(u,v)=4v^2u$. For $g(u,v)=f\left(x(u,v),y(u,v)\right)$, find the partial derivatives $\frac{\partial g}{\partial u}$ and $\frac{\partial g}{\partial v}$.

Solution

We first compute the partial derivatives $\frac{\partial f}{\partial x}=ye^{xy}$, $\frac{\partial f}{\partial y}=xe^{xy}$, $\frac{\partial x}{\partial u}=3\sin v$ and $\frac{\partial y}{\partial u}=4v^2$. The chain rule (Theorem 2) gives us

$$\frac{\partial g}{\partial u}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial u}=ye^{xy}\left(3\sin v\right)+xe^{xy}\left(4v^2\right).$$

Substituting for x and y, we get

$$\frac{\partial g}{\partial u}=12uv^2\sin v\ e^{12u^2v^2\sin v}+12uv^2\sin v\ e^{12u^2v^2\sin v}$$
$$=24uv^2\sin v\ e^{12u^2v^2\sin v}.$$

For the partial derivative of g with respect to v, we compute $\frac{\partial x}{\partial v}=3u\cos v$ and $\frac{\partial y}{\partial v}=8uv$. Here, the chain rule gives us :

$$\frac{\partial g}{\partial v}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial v}=ye^{xy}\left(3u\cos v\right)+xe^{xy}\left(8uv\right).$$

Substituting for x and y, we have : $\frac{\partial g}{\partial v}=\left(12u^2v^2\cos v+24u^2v\sin v\right)e^{12u^2v^2\sin v}$.

Example 3 (Converting from Rectangular to Polar Coordinates)

For a differentiable function $f(x, y)$ with $x = r\cos\theta$ and $y = r\sin\theta$, show that $f_r = f_x\cos\theta + f_y\sin\theta$ and $f_{rr} = f_{xx}\cos^2\theta + 2f_{xy}\cos\theta\sin\theta + f_{yy}\sin^2\theta$.

Solution

First, notice that $\frac{\partial x}{\partial r} = \cos\theta$ and $\frac{\partial y}{\partial r} = \sin\theta$. From Theorem 2, we now have

$$f_r = f_x\frac{\partial x}{\partial r} + f_y\frac{\partial y}{\partial r} = f_x\cos\theta + f_y\sin\theta .$$

Be very careful when computing the second partial derivative. Using the expression we have already found for f_r and Theorem2, we have

$$\begin{aligned}
f_{rr} &= \frac{\partial}{\partial r}(f_r) = \frac{\partial}{\partial r}\left(f_x\cos\theta + f_y\sin\theta\right)\\
&= \frac{\partial}{\partial r}\left(f_x\cos\theta\right) + \frac{\partial}{\partial r}\left(f_y\sin\theta\right)\\
&= \left[\frac{\partial}{\partial x}(f_x)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}(f_x)\frac{\partial y}{\partial r}\right]\cos\theta + \left[\frac{\partial}{\partial x}\left(f_y\right)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(f_y\right)\frac{\partial y}{\partial r}\right]\sin\theta\\
&= \left[f_{xx}\cos\theta + f_{xy}\sin\theta\right]\cos\theta + \left[f_{yx}\cos\theta + f_{yy}\sin\theta\right]\sin\theta\\
&= f_{xx}\cos^2\theta + 2f_{xy}\sin\theta\cos\theta + f_{yy}\sin^2\theta.
\end{aligned}$$

Implicit Differentiation

- Suppose that the equation $F(x, y) = 0$ defines y implicitly as a function of x, say $y = f(x)$. We let $z = F(x, y)$, where $x = t$ and $y = f(t)$. From Theorem1, we have $\frac{dz}{dt} = F_x\frac{dx}{dt} + F_y\frac{dy}{dt}$**.** But, since $z = F(x, y) = 0$, we have $\frac{dz}{dt} = 0$. Further, since $x = t$, we have $\frac{dx}{dt} = 1$ and $\frac{dy}{dt} = \frac{dy}{dx}$. This gives us $0 = F_x + F_y\frac{dy}{dx}$. Notice that we can solve this for $\frac{dy}{dx}$, provided $F_y \neq 0$. In this case, we have : $\frac{dy}{dx} = -\frac{F_x}{F_y}$.
- Suppose that the equation $F(x, y, z) = 0$ implicitly defines a function $z = f(x, y)$, where f is differentiable. Then, we can find the partial derivatives f_x and f_y using the chain rule, as follows. We first let $w = F(x, y, z)$. From the chain rule, we have $\frac{\partial w}{\partial x} = F_x\frac{\partial x}{\partial x} + F_y\frac{\partial y}{\partial x} + F_z\frac{\partial z}{\partial x}$. Notice that since $w = F(x, y, z) = 0$, $\frac{\partial w}{\partial x} = 0$. Also, $\frac{\partial x}{\partial x} = 1$ and $\frac{\partial y}{\partial x} = 0$, since x and y are

independent variables. This gives us $0 = F_x + F_z \frac{\partial z}{\partial x}$**.** We can solve this for $\frac{\partial z}{\partial x}$, as long as $F_z \neq 0$, to obtain: $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$.

Likewise, differentiating w with respect to y leads us to: $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$, $F_z \neq 0$.

Example 4 (Finding Partial Derivatives Implicitly)

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, given that $F(x, y, z) = xy^2 + z^3 + \sin(xyz) = 0$.

Solution

First, note that using the usual chain rule, we have: $F_x = y^2 + yz\cos(xyz)$,

$F_y = 2xy + xz\cos(xyz)$ and $F_z = 3z^2 + xy\cos(xyz)$.

If $3z^2 + xy\cos(xyz) \neq 0$ then

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{y^2 + yz\cos(xyz)}{3z^2 + xy\cos(xyz)} \text{ and } \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{2xy + xz\cos(xyz)}{3z^2 + xy\cos(xyz)}.$$

12.9 The gradient and Directional derivatives

In this section, we develop the notion of directional derivatives. Suppose that we want to find the instantaneous rate of change of $f(x, y)$ at the point $P(a,b)$ and in the direction given by the *unit* vector $u = \langle u_1, u_2 \rangle$. Let $Q(x, y)$ be any point on the line through $P(a,b)$ in the direction of u . Notice that the vector $\overrightarrow{PQ}$ is then parallel to u . Since two vectors are parallel if and only if one is a scalar multiple of the other, we have that $\overrightarrow{PQ} = h.u$, for some scalar h , so that $\overrightarrow{PQ} = \langle x-a, y-b \rangle = hu = h\langle u_1, u_2 \rangle = \langle hu_1, hu_2 \rangle$.

It then follows that $x - a = hu_1$ and $y - b = hu_2$, so that $x = a + hu_1$ and $y = b + hu_2$.

The point Q is then described by $(a + hu_1, b + hu_2)$, as indicated in Figure 1. Notice that the average rate of change of $z = f(x, y)$ along the line from P to Q is then

$$\frac{f(a + hu_1, b + hu_2) - f(a,b)}{h}.$$

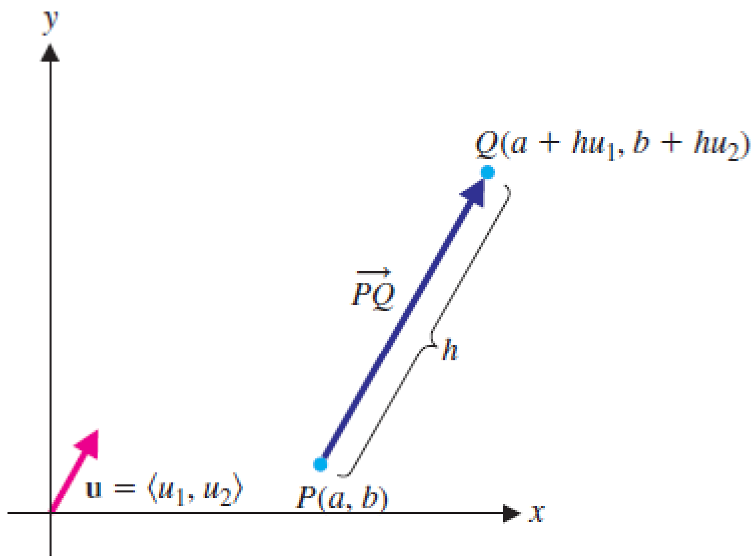


Figure1: The vector $\overrightarrow{PQ}$.

The instantaneous rate of change of $f\ (x, y)$ at the point $P(a,b)$ and in the direction of the unit vector u is then found by taking the limit as $h \to 0$.

Definition1

*The **directional derivative of** $f\ (x, y)$ at the point (a,b) and in the direction of the unit vector* $u = < u_1, u_2 >$ *is given by* $D_u\ f\ (a,b) = \lim_{h \to 0} \frac{f\ (a + hu_1, b + hu_2) - f\ (a,b)}{h}$, *provided the limit exists*.

Remark1:

We can extend the definition of the directional derivative of a function in 3 variables as:
The **directional derivative of** $f\ (x, y, z)$ at the point (a,b,c) and in the direction of the unit vector $u = < u_1, u_2, u_3 >$ is given by

$D_u\ f\ (a,b,c) = \lim_{h \to 0} \frac{f\ (a + hu_1, b + hu_2, c + hu_3) - f\ (a,b,c)}{h}$, provided the limit exists.

<u>Theorem1</u>

- Suppose that f is differentiable at (a,b) and $u = < u_1, u_2 >$ is any unit vector. Then, we can write $D_u\ f = f_x(a,b)u_1 + f_y(a,b)u_2$.

- Suppose that f is differentiable at (a,b,c) and $u = < u_1, u_2, u_3 >$ is any unit vector. Then, we can write $D_u\ f = f_x(a,b,c)u_1 + f_y(a,b,c)u_2 + f_z(a,b,c)u_3$.

Example 1 (Computing Directional Derivatives)

For $f(x,y) = x^2 y - 4y^3$, compute $D_u f(2,1)$ for the directions

(a) $u = < \frac{\sqrt{3}}{2}, \frac{1}{2} >$

(b) u in the direction from $(2,1)$ to $(4,0)$.

Solution

Regardless of the direction, we first need to compute the first partial derivatives $\frac{\partial f}{\partial x} = 2xy$ and $\frac{\partial f}{\partial y} = x^2 - 12y^2$. Then, $f_x(2,1) = 4$ and $f_y(2,1) = -8$.

- For (a), the unit vector is given as $u = < \frac{\sqrt{3}}{2}, \frac{1}{2} >$ and so, from Theorem 1 we have $D_u f(2,1) = f_x(2,1)u_1 + f_y(2,1)u_2 = 4\frac{\sqrt{3}}{2} - 8\frac{1}{2} = 2\sqrt{3} - 4 < 0$. Notice that this says that the function is decreasing in this direction.
- For (b), we must first find the unit vector u in the indicated direction. Observe that the vector from $(2,1)$ to $(4,0)$ corresponds to the position vector $< 2, -1 >$ and so, the unit vector in that direction is $u = \frac{< 2, -1 >}{\| < 2, -1 > \|} = < \frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}} >$. We then have from Theorem 1 that $D_u f(2,1) = f_x(2,1)u_1 + f_y(2,1)u_2 = 4\frac{2}{\sqrt{5}} + (-8)\frac{(-1)}{\sqrt{5}} = \frac{16}{\sqrt{5}} > 0$. So, the function is increasing rapidly in this direction.

For convenience, we define the **gradient** of a function to be the *vector-valued function* whose components are the first-order partial derivatives of f. We denote the gradient of a function f by **grad** f or ∇f.

Definition 2

*The **gradient** of $f(x,y)$ is the vector-valued function*

$\nabla f(x,y) = < \frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y) > = \frac{\partial f}{\partial x}(x,y)\vec{i} + \frac{\partial f}{\partial y}(x,y)\vec{j}$, *provided both partial derivatives exist. Similarly, we define the gradient of $f(x,y,z)$ as the vector-valued function*

$\nabla f(x,y,z) = < \frac{\partial f}{\partial x}(x,y,z), \frac{\partial f}{\partial y}(x,y,z), \frac{\partial f}{\partial z}(x,y,z) > = \frac{\partial f}{\partial x}(x,y,z)\vec{i} + \frac{\partial f}{\partial y}(x,y,z)\vec{j} + \frac{\partial f}{\partial z}(x,y,z)\vec{k}$,

provided all the partial derivatives are defined.

Theorem 2

If f is a differentiable function of x and y and u is any unit vector, then

$D_u f(x,y) = \nabla f(x,y).u$

Similarly, if f is a differentiable function of x , y and z and u is any unit vector, then $D_u f(x,y,z)=\nabla f(x,y,z).u$

Example 2 (Finding Directional Derivatives)
For $f(x,y)=x^2+y^2$, find $D_u f(1,-1)$ for
(a) u in the direction of $v=<-3,4>$.
(b) u in the direction of $v=<3,-4>$.

Solution

First, note that $\nabla f(x,y)=<\frac{\partial f}{\partial x}(x,y),\frac{\partial f}{\partial y}(x,y)>=<2x,2y>$.

At the point $(1,-1)$, we have $\nabla f(1,-1)=<2,-2>$.

- For (a), a unit vector in the same direction as v is $u=<\frac{-3}{5},\frac{4}{5}>$. The directional derivative of f in this direction at the point $(1,-1)$ is then
  $D_u f(1,-1)=<2,-2>.<\frac{-3}{5},\frac{4}{5}>=2\times\frac{-3}{5}+(-2)\times\frac{4}{5}=\frac{-14}{5}$.
- For (b), the unit vector is $u=<\frac{3}{5},\frac{-4}{5}>$ and so, the directional derivative of f in this direction at $(1,-1)$ is $D_u f(1,-1)=<2,-2>.<\frac{3}{5},\frac{-4}{5}>=2\times\frac{3}{5}+(-2)\times\frac{-4}{5}=\frac{14}{5}$.

<u>**Theorem 3**</u>
Suppose that f is a differentiable function of x and y at the point (a,b). Then

- **the maximum rate of change of f at (a,b) is $\|\nabla f(a,b)\|$, occurring in the direction of the gradient;**
- **the minimum rate of change of f at (a,b) (a, b) is $-\|\nabla f(a,b)\|$, occurring in the direction opposite the gradient;**
- **the rate of change of f at (a,b) is 0 in the directions orthogonal to $\nabla f(a,b)$.**
- **the gradient $\nabla f(a,b)$ is orthogonal to the level curve $f(x,y)=c$ at the point (a,b), where $c=f(a,b)$.**

Example 3 (Finding Maximum and Minimum Rates of Change)

Find the maximum and minimum rates of change of the function $f(x,y)=x^2+y^2$ at the point $(1,3)$.

Solution

We first compute the gradient $\nabla f = \langle 2x, 2y \rangle$ and evaluate it at the point $(1,3)$; $\nabla f(1,3) = \langle 2,6 \rangle$. From Theorem 3, the maximum rate of change of f at $(1,3)$ is $\|\nabla f(1,3)\| = \sqrt{40} = 2\sqrt{10}$ and occurs in the direction of $u = \frac{\nabla f(1,3)}{\|\nabla f(1,3)\|} = \langle \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \rangle$.

Similarly, the minimum rate of change of f at $(1,3)$ is $-\|\nabla f(1,3)\| = -\sqrt{40} = -2\sqrt{10}$, which occurs in the direction of $u = -\frac{\nabla f(1,3)}{\|\nabla f(1,3)\|} = \langle \frac{-1}{\sqrt{10}}, \frac{-3}{\sqrt{10}} \rangle$.

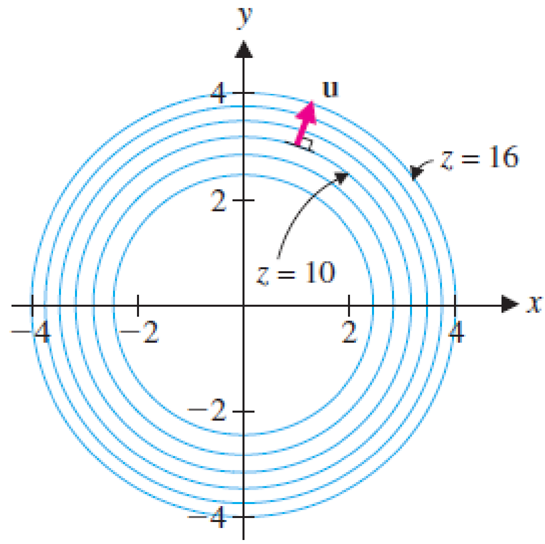


Figure2: Contour Plot of $z = x^2 + y^2$.

Example 4 (Finding the Direction of Maximum Increase)

If the temperature at point (x, y, z) is given by $T(x, y, z) = 85 + \left(1 - \frac{z}{100}\right)e^{-(x^2+y^2)}$, find the direction from the point $(2,0,99)$ in which the temperature increases most rapidly.

Solution

We first compute the gradient

$$\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle$$

$$= \langle -2x\left(1 - \frac{z}{100}\right)e^{-(x^2+y^2)}, -2y\left(1 - \frac{z}{100}\right)e^{-(x^2+y^2)}, \frac{-1}{100}e^{-(x^2+y^2)} \rangle$$

and $\nabla f(2,0,99) = \langle \frac{-1}{25}e^{-4}, 0, \frac{-1}{100}e^{-4} \rangle$. To find a unit vector in this direction, you can simplify the algebra by canceling the common factor of e^{-4} and multiplying by 100. A unit vector in the direction of $< -4, 0, -1 >$ and also in the direction of $\nabla f\ (2,0,99)$ is then $< \frac{-4}{\sqrt{17}}, 0, \frac{-1}{\sqrt{17}} >$.

Theorem 4

Suppose that $f(x,y,z)$ has continuous partial derivatives at the point (a,b,c) and $\nabla f\ (a,b,c) \neq 0$. Then, $\nabla f\ (a,b,c)$ is a normal vector to the tangent plane to the surface $f(x,y,c) = k$, at the point (a,b,c). Further, the equation of the tangent plane is $f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c) = 0$.

Example 5 (Using a Gradient to Find a Tangent Plane and Normal Line to a Surface)

Find equations of the tangent plane and the normal line to $x^3y - y^2 + z^2 = 7$ at the point $(1,2,3)$.

Solution

If we interpret the surface as a level surface of the function $f(x,y,z) = x^3y - y^2 + z^2$, a normal vector to the tangent plane at the point $(1,2,3)$ is given by $\nabla f\ (1,2,3)$. We have $\nabla f = < 3x^2y, x^3 - 2y, 2z >$ and $\nabla f\ (1,2,3) = < 6, -3, 6 >$. Given the normal vector $< 6, -3, 6 >$ and point $(1,2,3)$, an equation of the tangent plane is

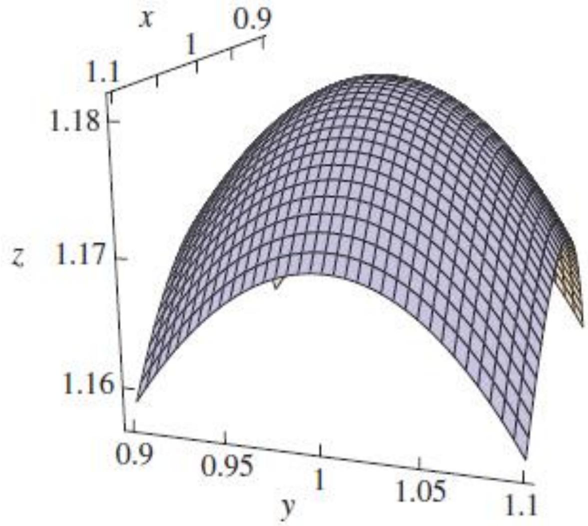
$$6(x-1) - 3(y-2) + 6(z-3) = 0 \ .$$

The normal line has parametric equations $\begin{cases} x = 1 + 6t \\ y = 2 - 3t \\ z = 3 + 6t \end{cases}, t \in \mathbb{R}.$

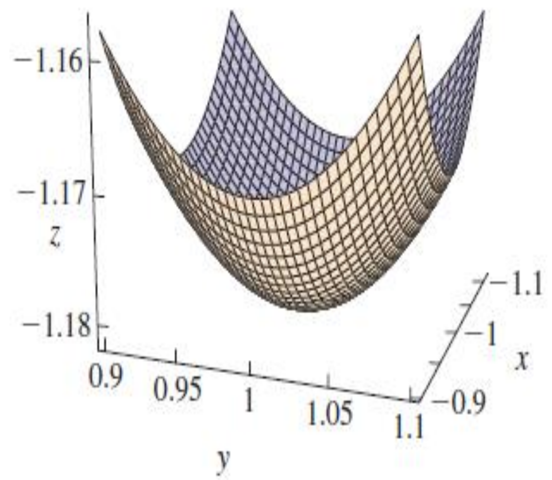
12.10 Extrema of functions of several variables

Definition1

*We call $f(a,b)$ a **local maximum** of f if there is an open disk R centered at (a,b), for which $f(a,b) \geq f(x,y)$ for all $(x,y) \in R$. Similarly, $f(a,b)$ is called a **local minimum** of f if there is an open disk R centered at (a,b), for which $f(a,b) \leq f(x,y)$ for all $(x,y) \in R$. In either case, $f(a,b)$ is called a **local extremum** of f.*



Local maximum at $(1,1)$ for $f(x,y) = x\,e^{-\frac{x^2}{2}-\frac{y^3}{3}+y}$



Local minimum at (-1.1) for $f(x,y) = x\,e^{-\frac{x^2}{2}-\frac{y^3}{3}+y}$

Definition2

*The point (a,b) is a **critical point** of the function $f(x,y)$ if (a,b) is in the domain of f and either $\frac{\partial f}{\partial x}(a,b) = \frac{\partial f}{\partial y}(a,b) = 0$ or one or both of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ do not exist at (a,b).*

Theorem1

If $f(x,y)$ has a local extremum at (a,b), then (a,b) must be a critical point of f.

Example1

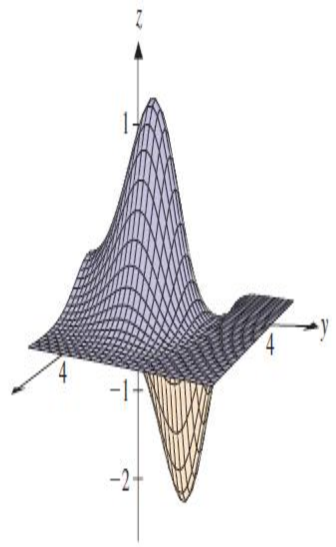
Find all critical points of $f(x,y) = x\,e^{-\frac{x^2}{2}-\frac{y^3}{3}+y}$.

Solution

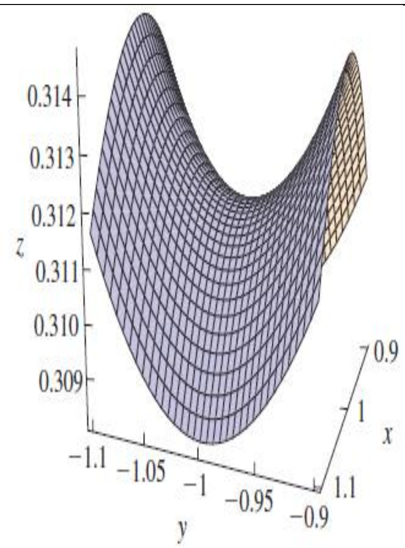
First, we compute the first partial derivatives:

$$\frac{\partial f}{\partial x}(x,y) = (1-x^2)\,e^{-\frac{x^2}{2}-\frac{y^3}{3}+y} \text{ and } \frac{\partial f}{\partial y}(x,y) = x(1-y^2)\,e^{-\frac{x^2}{2}-\frac{y^3}{3}+y}.$$

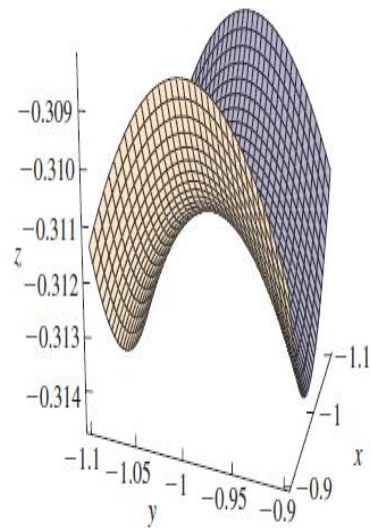
Since exponentials are always positive, we have $\frac{\partial f}{\partial x}(x,y)=0$ if and only if $1-x^2=0$, that is, when $x=\pm 1$. We have $\frac{\partial f}{\partial y}(x,y)=0$ if and only if $x\left(1-y^2\right)=0$, that is, when $x=0$ or $y=\pm 1$. So the set of critical points is $C_f=\left\{(-1,-1),(-1,1),(1,-1),(1,1)\right\}$.



$z=f(x,y)=x\,e^{-\frac{x^2}{2}-\frac{y^3}{3}+y}$



Saddle point at $\left(1,-1\right)$.



Saddle point at $\left(-1,-1\right)$.

Definition3

*The point $P\left(a,b,f(a,b)\right)$ is a **saddle point** of $z=f(x,y)$ if (a,b) is a critical point of f and if every open disk centered at (a,b) contains points (x,y) in the domain of f for which $f(x,y)<f(a,b)$ and points (x,y) in the domain of f for which $f(x,y)>f(a,b)$.*

Theorem2 (Second Derivatives Test)

Suppose that $f(x,y)$ has continuous second-order partial derivatives in some open disk containing the point (a,b) and that $f_x(a,b)=f_y(a,b)=0$. Define the discriminant D for the point (a,b) by $D(a,b)=f_{xx}(a,b)f_{yy}(a,b)-\left[f_{xy}(a,b)\right]^2$.

- **If $D(a,b)>0$ and $f_{xx}(a,b)>0$, then *f* has a local minimum at (a,b).**
- **If $D(a,b)>0$ and $f_{xx}(a,b)<0$, then *f* has a local maximum at (a,b).**
- **If $D(a,b)<0$, then *f* has a saddle point at (a,b).**
- **If $D(a,b)=0$, then no conclusion can be drawn.**

Example2 (Using the Discriminant to Find Local Extrema)

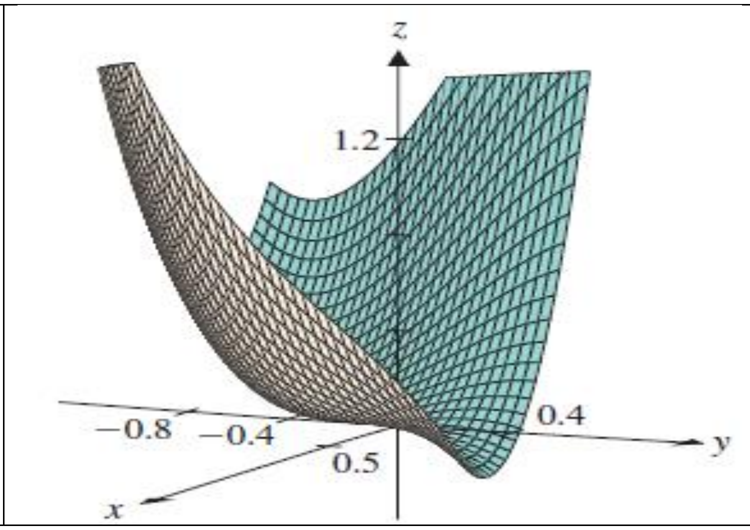
Locate and classify all critical points for $f(x, y) = 2x^2 - y^3 - 2xy$.

Solution

We first compute the first partial derivatives: $f_x = 4x - 2y$ and $f_y = -3y^2 - 2x$. Since both f_x and f_y are defined for all (x, y) , the critical points are solutions of the two equations: $f_x = 4x - 2y = 0$ and $f_y = -3y^2 - 2x = 0$. Solving the first equation for y , we get $y = 2x$. Substituting this into the second equation, we have $0 = -3(4x^2) - 2x = -12x^2 - 2x = -2x(6x + 1)$, so that $x = 0$ or $x = \frac{-1}{6}$. The corresponding y -values are $y = 0$ and $y = \frac{-1}{3}$. The only two critical points are then $(0,0)$ and $\left(\frac{-1}{6}, \frac{-1}{3}\right)$. To classify these points, we first compute the second partial derivatives: $f_{xx} = 4, f_{yy} = -6y$ and $f_{xy} = -2$, and then test the discriminant. We have $D(0,0) = 4 \times 0 - (-2)^2 = -4 < 0$ and $D\left(\frac{-1}{6}, \frac{-1}{3}\right) = 4 \times (-6) \times \left(\frac{-1}{3}\right) - (-2)^2 = 4 > 0$.

From Theorem 2, we conclude that there is a <u>*saddle point*</u> *of* f *at* $(0,0)$, since $D(0,0) < 0$. Further, there is a <u>*local minimum at*</u> $\left(\frac{-1}{6}, \frac{-1}{3}\right)$ since $D\left(\frac{-1}{6}, \frac{-1}{3}\right) > 0$ and $f_{xx}\left(\frac{-1}{6}, \frac{-1}{3}\right) = 4 > 0$.

Point	$(0, 0)$	$\left(-\frac{1}{6}, -\frac{1}{3}\right)$
$f_{xx} = 4$	4	4
$f_{yy} = -6y$	0	2
$f_{xy} = -2$	−2	−2
$D(a, b)$	−4	4

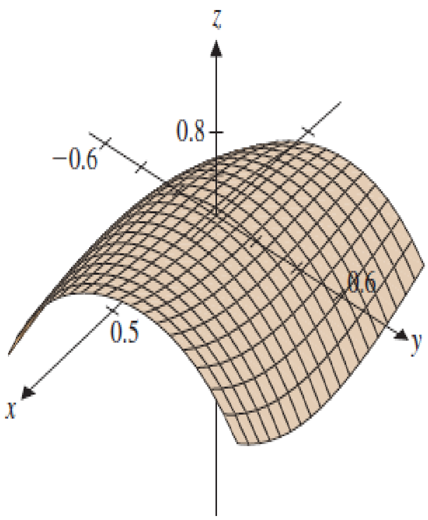


Example3 (Classifying Critical Points)

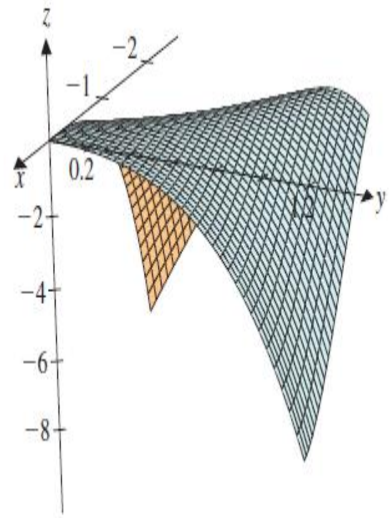
Locate and classify all critical points for $f(x,y) = x^3 - 2y^2 - 2y^4 + 3x^2 y$.

Solution

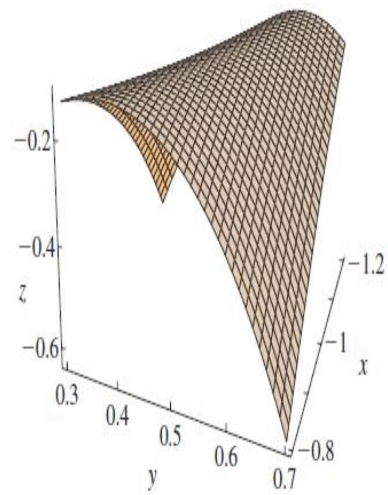
Here, we have $f_x = 3x^2 + 6xy$ and $f_y = -4y - 8y^3 + 3x^2$. Since both f_x and f_y exist for all (x,y), the critical points are solutions of the two equations: $f_x = 3x^2 + 6xy = 0$ and $f_y = -4y - 8y^3 + 3x^2$=0. From the first equation, we have $0 = 3x^2 + 6xy = 3x(x + 2y)$, so that at a critical point, $x = 0$ or $x = -2y$.
Substituting $x = 0$ into the second equation, we have $0 = -4y - 8y^3 = -4y(1 + 2y^2)$.
The only (real) solution of this equation is $y = 0$. This says that for $x = 0$, we have only one critical point: $(0,0)$.
Substituting $x = -2y$ into the second equation, we have $0 = -4y - 8y^3 + 3(-2y)^2 = -4y(1 + 2y^2 - 3y) = -4y(2y - 1)(y - 1)$. The solutions of this equation are $y = 0, y = \frac{1}{2}$ and $y = 1$, with corresponding critical points $(0,0), (-1, \frac{1}{2})$ and $(-2,1)$.
To classify the critical points, we compute the second partial derivatives,
$f_{xx} = \frac{\partial}{\partial x}\left(3x^2 + 6xy\right) = 6x + 6y$ $f_{yy} = \frac{\partial}{\partial y}\left(-4y - 8y^3 + 3x^2\right) = -4 - 24y^2$, and $f_{xy} = \frac{\partial}{\partial y}\left(3x^2 + 6xy\right) = 6x$, and evaluate the discriminant at each critical point. We have $D(0,0) = 0$, $D\left(-1, \frac{1}{2}\right) = -6 < 0$ and $D(-2,1) = 24 > 0$. From Theorem 2, we conclude that <u>*f* has a saddle point</u> at $\left(-1, \frac{1}{2}\right)$, since $D\left(-1, \frac{1}{2}\right) = -6 < 0$. Further, <u>*f* has a local maximum at</u> $(-2,1)$ since $D(-2,1) = 24 > 0$ and $f_{xx}(-2,1) = -3 < 0$. Unfortunately, Theorem 2 gives us no information about the critical point $(0,0)$, since $D(0,0) = 0$.
However, notice that in the plane $y = 0$ we have $f(x,y) = x^3$. In two dimensions, the curve $z = x^3$ has an inflection point at $x = 0$. This shows that there is <u>no local extremum at</u> $(0,0)$.



The surface near $(0,0)$



The surface near $(-2,1)$



The surface near $\left(-1,\frac{1}{2}\right)$

Definition4

*We call $f(a,b)$ the **absolute maximum** of f on the region R if $f(a,b) \geq f(x,y)$ for all $(x,y) \in R$. Similarly, $f(a,b)$ is called the **absolute minimum** of f on R if $f(a,b) \leq f(x,y)$ for all $(x,y) \in R$. In either case, $f(a,b)$ is called an **absolute extremum** of f.*

<u>Theorem 3 (Extreme Value Theorem)</u>

Suppose that $f(x,y)$ is continuous on the closed and bounded region $R \subset \mathbb{R}^2$. Then f has both an absolute maximum and an absolute minimum on R. Further, an absolute extremum may only occur at a critical point in R or at a point on the boundary of R.

12.11 Constrained Optimization and Lagrange Multipliers

In this section, we develop a technique for finding the maximum or minimum of a function, given one or more constraints on the function's domain.

<u>Theorem1</u>

Suppose that $f(x,y,z)$ and $g(x,y,z)$ are functions with continuous first partial derivatives and $\nabla g(x,y,z) \neq 0$ on the surface $g(x,y,z)=0$. Suppose that either the minimum (or the maximum) value of $f(x,y,z)$ subject to the constraint $g(x,y,z)=0$ occurs at (x_0,y_0,z_0). Then $\nabla f(x_0,y_0,z_0)=\lambda \nabla g(x_0,y_0,z_0)$, for some constant λ (called a Lagrange multiplier).

Remark1

- Note that Theorem 1 says that if $f(x,y,z)$ has an extremum at a point (x_0,y_0,z_0) on the surface $g(x,y,z)=0$, we will have for $(x,y,z)=(x_0,y_0,z_0)$,

$$\begin{cases} f_x(x,y,z)=\lambda g_x(x,y,z) \\ f_y(x,y,z)=\lambda g_y(x,y,z) \\ f_z(x,y,z)=\lambda g_z(x,y,z) \\ g(x,y,z)=0 \end{cases}$$

  Finding such extrema then boils down to solving these four equations for the four unknowns x,y,z and λ .

- Notice that the Lagrange multiplier method we have just developed can also be applied to functions of two variables, by ignoring the third variable in Theorem1. That is, if $f(x,y)$ and $g(x,y)$ have continuous first partial derivatives and $f(x_0,y_0)$ is an extremum of f , subject to the constraint $g(x,y)=0$, then we must have $\nabla f(x_0,y_0)=\lambda\nabla g(x_0,y_0)$,for some constant λ .In this case, we end up with the three equations $f_x(x,y)=\lambda g_x(x,y)$, $f_y(x,y)=\lambda g_y(x,y)$ and $g(x,y)=0$, for the three unknowns x,y and λ.

Example 1 (Finding a Minimum Distance)

Use Lagrange multipliers to find the point on the line $y=3-2x$ that is closest to the origin.

Solution

For $f(x,y)=x^2+y^2$, we have $\nabla f(x,y)= <2x,2y>$ and for $g(x,y)=2x+y-3$, we have $\nabla g(x,y)= <2,1>$. The vector equation $\nabla f(x,y)=\lambda\nabla g(x,y)$ becomes $<2x,2y>=\lambda <2,1>$ from which it follows that $2x=2\lambda$ and $2y=\lambda$.
The second equation gives us $\lambda=2y$. The first equation then gives us $x=\lambda=2y$.
Substituting $x=2y$ into the constraint equation $y=3-2x$, we have $5y=3$.
The solution is $y=\frac{3}{5}$, giving us $x=2y=\frac{6}{5}$. The closest point is then $\left(\frac{6}{5},\frac{3}{5}\right)$.

Example 2 (Optimization with an Inequality Constraint)

Suppose that the temperature of a metal plate is given by $T(x,y)=x^2+2x+y^2$, for points (x,y) on the elliptical plate defined by $x^2+4y^2\le 24$. Find the maximum and minimum temperatures on the plate.

Solution
The plate corresponds to the shaded region R shown in Figure 1.

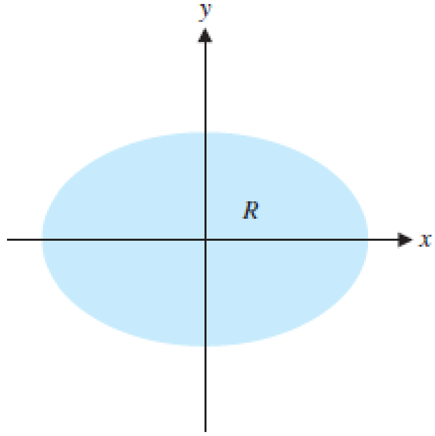


Figure 1: A metal plate.

We first look for critical points of $T(x,y)$ inside the region R. We have $\nabla T(x,y) = <2x+2, 2y> = <0,0>$ if $(x,y)=(-1,0)$, which is in R. At this point, $T(-1,0)=-1$. We next look for the extrema of $T(x,y)$ on the ellipse $x^2+4y^2=24$. We first rewrite the constraint equation as $g(x,y)=x^2+4y^2-24=0$. From Theorem 1, any extrema on the ellipse will satisfy the Lagrange multiplier equation: $\nabla T(x,y)=\lambda \nabla g(x,y)$ or $<2x+2,2y> = \lambda <2x,8y> = <2\lambda x, 8\lambda y>$.
This occurs when $2x+2=2\lambda x$ and $2y=8\lambda y$.

Notice that the second equation holds when $y=0$ or $\lambda = \frac{1}{4}$.

If $y=0$, the constraint $x^2+4y^2=24$ gives $x=\pm\sqrt{24}$.

If $\lambda=\frac{1}{4}$, the first equation becomes $2x+2=\frac{1}{2}x$ so that $x=-\frac{4}{3}$. The constraint $x^2+4y^2=24$ now gives $y=\pm\frac{\sqrt{50}}{3}$.

Finally, we compare the function values at all of these points (the one interior critical point and the candidates for boundary extrema):
and $T(-1,0)=-1$, $T(\sqrt{24},0)=24+\sqrt{24}\approx 33.8$, $T(-\sqrt{24},0)=24-2\sqrt{24\approx 14.2}$
$T\left(-\frac{4}{3},\frac{\sqrt{50}}{3}\right)=\frac{14}{3}\approx 4.7$, $T\left(-\frac{4}{3},-\frac{\sqrt{50}}{3}\right)=\frac{14}{3}\approx 4.7$.

From this list, it's easy to identify the minimum value of -1 at the point $(-1,0)$ and the maximum value of $24+2\sqrt{24}$ at the point $(\sqrt{24},0)$.

We close this section by considering the case of finding the minimum or maximum value of a differentiable function $f(x,y,z)$ subject to two constraints $g(x,y,z)=0$ and $h(x,y,z)=0$, where g and h are also differentiable (see Figure 2 below).

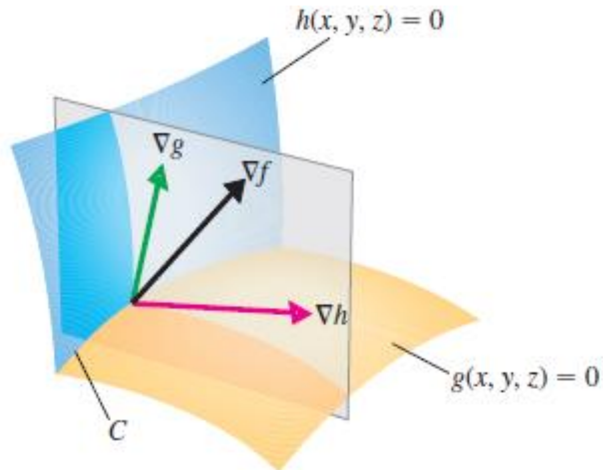


Figure 2: Constraint surfaces and the plane determined by the normal vectors ∇g and ∇h.

The method of Lagrange multipliers for the case of two constraints then consists of finding the point (x,y,z) and the Lagrange multipliers λ and μ (for a total of five unknowns) satisfying the five equations defined by:

$$\begin{cases} f_x(x,y,z)=\lambda\, g_x(x,y,z)+\mu\, h_x(x,y,z) \\ f_y(x,y,z)=\lambda\, g_y(x,y,z)+\mu\, h_y(x,y,z) \\ f_z(x,y,z)=\lambda\, g_z(x,y,z)+\mu\, h_z(x,y,z) \\ \quad g(x,y,z)=0 \quad \& \quad h(x,y,z)=0 \end{cases}.$$

Example 3 (Optimization with Two Constraints)
The plane $x+y+z=12$ intersects the paraboloid $z=x^2+y^2$ in an ellipse. Find the point on the ellipse that is closest to the origin.

Solution
We illustrate the intersection of the plane with the paraboloid in Figure 3.

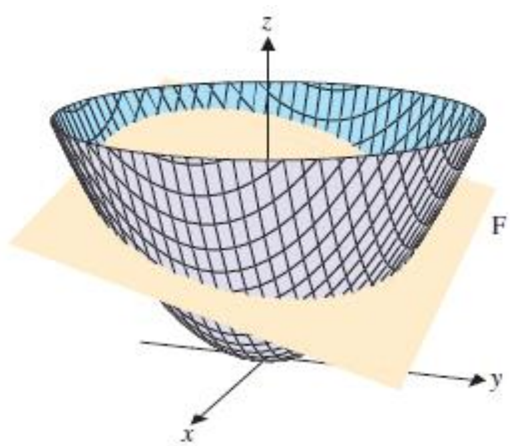


Figure 3: Intersection of a paraboloid and a plane.

Observe that minimizing the distance to the origin is equivalent to minimizing $f(x,y,z)=x^2+y^2+z^2$ [the *square* of the distance from the point (x,y,z) to the origin]. Further, the constraints may be written as $g(x,y,z)=x+y+z-12=0$ and $h(x,y,z)=x^2+y^2-z=0$. At any extremum, we must have that

$$\nabla f(x,y,z)=\lambda\nabla g(x,y,z)+\mu\nabla h(x,y,z) \text{ or}$$
$$<2x,2y,2z>=\lambda<1,1,1>+\mu<2x,2y,-1> .$$

Together with the constraint equations, we now have the system of equations:

$$\begin{cases} 2x=\lambda+2\mu x & (1) \\ 2y=\lambda+2\mu y & (2) \\ 2z=\lambda-\mu & (3) \\ x+y+z-12=0 \quad (4) \quad \& \quad x^2+y^2-z=0 & (5) \end{cases}$$

From (1), we have $\lambda=2x(1-\mu)$, while from (2), we have $\lambda=2y(1-\mu)$.
Setting these two expressions for λ equal gives us $2x(1-\mu)=2y(1-\mu)$,
from which it follows that either $\mu=1$ (in which case $\lambda=0$) or $x=y$. However, if $\mu=1$ and $\lambda=0$, we have from (3) that $z=-12$, which contradicts (5).
Consequently, the only possibility is to have $x=y$, from which it follows from (5) that $z=2x^2$. Substituting this into (4) gives us:
$0=x+y+z-12=x+x+2x^2-12=2x^2+2x-12=2(x+3)(x-2)$, so that $x=-3$ or $x=2$. Since $y=x$ and $z=2x^2$, we have that $(2,2,8)$ and $(-3,-3,18)$ are the only candidates for extrema. Finally, since $f(2,2,8)=72$ and $f(-3,-3,18)=342$, the closest point on the intersection of the two surfaces to the origin is $(2,2,8)$. By the same reasoning, observe that the farthest point on the intersection of the two surfaces from the origin is $(-3,-3,18)$.