(Part 2)

suppose that there is a function $f(x,y)$
where $x = r\cos\theta$
$y = r\sin\theta$
then

$$f_r = f_x \cos\theta + f_y \sin\theta$$

$$f_{rr} = f_{xx}\cos^2\theta + 2 f_{xy}\cos\theta\sin\theta + f_{yy}\sin^2\theta.$$

Proof

Notice that $\frac{dx}{dr} = \cos\theta$

$\frac{dy}{dr} = \sin\theta$

So, $f_r = f_x \frac{dx}{dr} + f_y \frac{dy}{dr} = f_x\cos\theta + f_y\sin\theta$

Now $f_{rr} = \frac{d}{dr}(f_r) = \frac{d}{dr}(f_x\cos\theta + f_y\sin\theta)$

$= \frac{d}{dr}(f_x\cos\theta) + \frac{d}{dr}(f_y\sin\theta) \quad (*)$

Now $\frac{d}{dr}(f_x\cos\theta) = \left[\frac{\partial}{\partial x}(f_x)\frac{dx}{dr} + \frac{\partial}{\partial y}(f_x)\frac{dy}{dr}\right]\cos\theta$

$= [f_{xx}\cos\theta + f_{xy}\sin\theta]\cos\theta.$

Similarly $\frac{d}{dr}(f_y\sin\theta) = \left[\frac{d}{dx}(f_y).\frac{dx}{dr} + \frac{d}{dy}(f_y)\frac{dy}{dr}\right]\sin\theta$

$= [f_{xy}\cdot\cos\theta + f_{yy}\sin\theta]\sin\theta$

substitute in $(*)$

$$f_{rr} = f_{xx}\cos^2\theta + 2f_{xy}\sin\theta\cos\theta + f_{yy}\sin^2\theta. \quad \blacksquare$$

Implicit Differentiation

Recall that if we have $x^2y^2 = 2xy$ and we need to calculate $\frac{dy}{dx} = y'$ then we would do the implicit differentiation as follows:

$$2xy^2\frac{dx}{dx} + x^2\left(2y\frac{dy}{dx}\right) = 2x\frac{dy}{dx} + 2\frac{dx}{dx}y$$

$$\Leftrightarrow \quad 2xy^2 + 2yx^2y' = 2xy' + 2y$$

$$\Leftrightarrow \quad (2yx^2 - 2x)\,y' = 2y - 2xy^2$$

$$\Leftrightarrow \quad y' = \frac{2y - 2xy^2}{2yx^2 - 2x} \quad \ldots$$

By the same manner we can do the implicite differentiation for $F(x,y,z) = 0$.

Rules:

If $f(x,y,z) = 0$ then

$$\frac{\partial z}{\partial x} = \frac{F_x}{F_z}$$

$$\frac{\partial z}{\partial y} = \frac{F_y}{F_z}$$

Example: Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ where

$$xy^2 + z^3 + \sin(xyz) = 0.$$

Solution

$F_x = y^2 + \cos(xyz)\cdot yz$

$F_y = 2yx + \cos(xyz)\cdot xz$

$F_z = 3z^2 + \cos(xyz)\cdot xy$

Therefore, $\frac{\partial z}{\partial x} = \frac{F_x}{F_z}$ and $\frac{\partial z}{\partial y} = \frac{F_y}{F_z}$

** Directional Derivative of $f(x,y)$

Definition suppose that $P(a,b)$ is a point Lies on $f(x,y)$ and $\vec{u}$ is the unit vector ~~with~~ which is parallel to $\vec{OP} = \langle a,b \rangle = \langle u_1, u_2 \rangle$.

Then the directional derivative at $P = (a,b)$ is given by

$$\boxed{\begin{aligned} D_u f(a,b) &= \lim_{h \to 0} \frac{f(a+hu_1, b+hu_2) - f(a,b)}{h} \\ &= f_x(a,b)\, u_1 + f_y(a,b)\, u_2 \end{aligned}}$$

Example :

suppose $f(x,y) = x^2 y - 4y^3$. Find $D_u f(2,1)$.

Solution

$$\vec{OP} = \langle 2,1 \rangle \Rightarrow u = \frac{\langle 2,1 \rangle}{\sqrt{4+1}} = \langle \overset{u_1}{\frac{2}{\sqrt{5}}}, \overset{u_2}{\frac{1}{\sqrt{5}}} \rangle$$

Now

$$f_x = 2xy \Rightarrow f_x(a,b) \cdot u_1 = 2(2)(1) \cdot \frac{2}{\sqrt{5}} = \frac{8}{\sqrt{5}}$$

$$f_y = x^2 - 12y^2 \Rightarrow f_y(a,b)\, u_2 = (2)^2 - 12(1)^2 \left(\frac{1}{\sqrt{5}}\right)$$

$$= 4 - \frac{12}{\sqrt{5}}$$

So, $D_u f(2,1) = \frac{8}{\sqrt{5}} + 4 - \frac{12}{\sqrt{5}} = 4 - \frac{4}{\sqrt{5}}$

** The gradient of the function $f(x,y)$:

$\nabla f(x,y) = \langle f_x , f_y \rangle$ is vector valued function

Remark

If f is differentiable function of x and y then

$$\boxed{D_u f(x,y) = \nabla f(x,y) \cdot u} \longrightarrow \text{Rule}$$

Example

Let $f(x,y) = x^2+y^2$. Find $D_u f(1,-1)$ for u is the direction of $v = \langle -3, 4\rangle$?

Solution :

$$u = \frac{v}{\|v\|} = \frac{\langle -3, 4\rangle}{\sqrt{9+16}} = \langle \frac{-3}{5}, \frac{4}{5} \rangle$$

So, $D_u f(x,y) = \nabla f(x,y) \cdot u$

$$= \langle f_x(1,-1), f_y(1,-1) \rangle \cdot \langle \frac{-3}{5}, \frac{4}{5} \rangle$$

(Notice that :
$f_x = 2x$
$f_y = 2y$)

$$= \langle 2(1), 2(-1) \rangle \cdot \langle \frac{-3}{5}, \frac{4}{5} \rangle$$

$$= \frac{-6}{5} - \frac{8}{5} = \frac{-14}{5}$$

Remark

Let $f(x,y)$ is differentiable at (a,b).

[1] The maximum rate of change of f at (a,b) is

$$\|\nabla f(a,b)\| = \|\langle f_x(a,b), f_y(a,b)\rangle\|$$

[2] The minimum rate of change of f at (a,b) is

$$-\|\nabla f(a,b)\|$$

[3] we say $\nabla f(a,b)$ is orthogonal to the Level curve $f(x,y) = c$ (where c is constant).

C is exactly equal to $f(a,b)$

Example:

Let $f(x,y) = x^2+y^2$. Find the maximum rate and minimum rate of changes of $f(x,y)$ at $(1,3)$?

Solution

$f_x = 2x \Rightarrow f_x(1,3) = 2(1) = 2$

$f_y = 2y \Rightarrow f_y(1,3) = 2(3) = 6$

$\nabla f(1,3) = \langle 2,6 \rangle$

$\| \nabla f(1,3) \| = \sqrt{4+36} = \sqrt{40}$

Therefore:

The maximum rate $= \sqrt{40}$

The minimum rate $= -\sqrt{40}$

Notice that:

If its required to find the direction which accurs to find maximum or minimum rate

$$\underbrace{\vec{u} = \frac{\nabla f(1,3)}{\| \nabla f(1,3) \|}}_{\text{For maximum rate}} \quad \text{or} \quad \underbrace{\vec{u} = \frac{\nabla f(1,3)}{-\| \nabla f(1,3) \|}}_{\text{For minimum rate}}$$

Rule: Let (a,b,c) be a point and $f(x,y,z)$ be a function. If $\nabla f(a,b,c) \neq 0$ then $\boxed{\nabla f(a,b,c) \text{ is a normal vector}}$ of the tangent plane of $f(x,y,z)$ at (a,b,c).

Example Let $(1,2,3)$ Lies on $f(x,y,z) = x^3y - y^2 + z^2$. Find the equation of tangent Plane at $(1,2,3)$?

Solution: The normal vector $= \vec{n} = \nabla f(1,2,3)$

$= \langle f_x(1,2,3), f_y(1,2,3), f_z(1,2,3) \rangle$

The Eqution of Plane $= \vec{n} \cdot \langle x-1, y-2, z-3 \rangle = 0$