Part 3

** Extrema of a function:

(1) $f(a,b)$ is a Local maximum if $\exists$ an open set $A \subseteq$ Domain of f such that $f(a,b) \geq f(x,y)$ for all $(x,y) \in A$

(2) By the same manner, we can define $f(a,b)$ is a Local minimum

(3) (a,b) is called a critical Point of $f(x,y)$ if is either $\frac{\partial f}{\partial x}(a,b) = \frac{\partial f}{\partial y}(a,b) = 0$

or

$\frac{\partial f}{\partial x}(a,b)$ or $\frac{\partial f}{\partial y}(a,b)$ is not existed

~~(4) If (a,b) is the center of an open set D (i.e. disc) such that f(x,y)~~

Example Find all critical Points of $f(x,y) = x e^{-\frac{x^2}{2} - \frac{y^3}{3} + y}$

Solution : we should compute :

$$f_x = e^{-\frac{x^2}{2} - \frac{y^3}{3} + y} + x e^{-\frac{x^2}{2} - \frac{y^3}{3} + y} \cdot (-x)$$

$$= (1 - x^2) e^{-\frac{x^2}{2} - \frac{y^3}{3} + y}$$

$$f_y = x(1 - y^2) e^{-\frac{x^2}{2} - \frac{y^3}{3} + y}$$

Notice that $f_x = 0 \iff (1-x^2) = 0 \iff x = \pm 1$

$f_y = 0 \iff x(1-y^2) = 0 \iff x = 0$ or $y = \pm 1$

So, $\{(1,1), (1,-1), (-1,1), (-1,-1)\}$ are critical points. (Notice that $x=0$ doesn't make $f_x = 0$).

There is no points make f_x or f_y is not existed $\blacksquare$

Def : Let (a,b) be a critical Point of $f(x,y)$ then the Point $(a,b,f(a,b))$ is called a saddle Point of $f(x,y)$.

Rules :

Let (a,b) be a Point such that $f_x(a,b) = f_y(a,b) = 0$

(1) The discriminate D for a Point (a,b) $= D(a,b) =$

$$f_{xx}(a,b) \cdot f_{yy}(a,b) - [f_{xy}(a,b)]^2.$$

(2) $D(a,b) > 0$ and $f_{xx}(a,b) > 0 \Rightarrow f(a,b)$ is Local minimum

$D(a,b) > 0$ and $f_{xx}(a,b) < 0 \Rightarrow f(a,b)$ is Local maximum

$D(a,b) < 0 \Rightarrow (a,b,f(a,b))$ is a saddle Point

$D(a,b) = 0 \Rightarrow$ we can not induce any thing anymore.

Example Find Local extrema of $f(x,y) = 2x^2 - y^3 - 2xy$

Solution we will use $D(a,b)$!!

$f_x = 4x - 2y$ and $f_y = -3y^2 - 2x$

Let $f_x = 0 \Rightarrow 4x - 2y = 0 \Rightarrow \boxed{y = 2x}$

if $f_y = 0 \Rightarrow -3y^2 - 2x = 0$

$\Rightarrow -3(2x)^2 - 2x = 0$

$\Rightarrow -12x^2 - 2x = 0$

$\Rightarrow \boxed{x = 0 \text{ or } x = \frac{-1}{6}} \Rightarrow \boxed{y = 0 \text{ or } \frac{-1}{3}}$

So, critical Points are $(0,0)$ or $(\frac{-1}{6}, \frac{-1}{3})$

Now we should compute :

$f_{xx} = 4$

$f_{yy} = -6y$

$f_{xy} = -2$

Now $D(0,0) = 4(0) - (-2)^2 = -4 < 0$

$D(\frac{-1}{6}, \frac{-1}{3}) = 4(-6) \times (\frac{-1}{3}) - (-2)^2 = 4 > 0$

Notice that

① $D(0,0) < 0 \Rightarrow (0, 0, f(0,0))$ is a saddle Point.

② $D(\frac{-1}{6}, \frac{-1}{3}) > 0$ and $f_{xx}(\frac{-1}{6}, \frac{-1}{3}) = 4 > 0 \Rightarrow$

$(\frac{-1}{6}, \frac{-1}{3}, f(\frac{-1}{6}, \frac{-1}{3}))$ is Local minimum

Remark

Let $f(x,y)$ is continous on the closed set $A \Rightarrow f$ has absolute extremum (i.e. minimum and maximum).

Notice that The absolute extremum accurs on the critical Points or on the boundry of A.

Remmark If $f(x,y,z)$ has extremum at (x_0, y_0, z_0) on the surface $g(x,y,z) = 0$ Then

$$\begin{cases} f_x(x_0, y_0, z_0) = \lambda\, g_x(x_0, y_0, z_0) \\ f_y(x_0, y_0, z_0) = \lambda\, g_y(x_0, y_0, z_0) \\ f_z(x_0, y_0, z_0) = \lambda\, g_z(x_0, y_0, z_0) \\ g(x_0, y_0, z_0) = 0 \end{cases}$$

$\boxed{\lambda \text{ is called Lagrange multiplier}}$

Example

Finding the minimum distance

Use Lagrange multipliers to find to point on the Line $y = 3-2x$ that is closet to the origin?

Solution: suppose that $f(x,y) = x^2+y^2$.

(in $\mathbb{R}^3$) Centered by $(0,0)$

$\nabla f(x,y) = \langle 2x, 2y \rangle$

Let ~~g(x,y)=~~ $g(x,y) = 2x+y-3$

$\nabla g(x,y) = \langle 2, 1 \rangle$

Now

$\nabla f(x,y) = \lambda \nabla g(x,y)$

$\Rightarrow 2x = 2\lambda$ and $2y = \lambda$

$\Rightarrow \boxed{x = \lambda = 2y}$

Now

as $g(x,y) = 2x+y-3$

$= 4y+y-3$ $\cancel{= 2x+2x-3}$ (since $2y = x$)

$= 5y-3$ $\cancel{= 4x-3}$

If $g(x,y) = 0 \Rightarrow y = \frac{3}{5}$ and

$x = \frac{6}{5}$

So, $\left(\frac{6}{5}, \frac{3}{5}\right)$ is the closet point.

Example

optimization with an inequality Constraint !!

suppose $T(x,y) = x^2+2x+y^2$, for a point (x,y) on $x^2+4y^2 \leq 24$. Find the maximum and minimum of $T(x,y)$?

Solution: Firstly, we should compute critical Point of $T(x,y)$:

$f_x = 2x+2$ and $f_y = 2y$

If $f_x = f_y = 0$ then $(-1,0)$ is the critical Point.

secondly, we compute the extrema of $T(x,y)$ on $x^2+4y^2=24$:

STEP 1 $\quad g(x,y) = x^2+4y^2-24 = 0$

STEP 2 $\quad \nabla T(x,y) = \langle 2x+2 , 2y \rangle$

$\nabla g(x,y) = \langle 2x , 8y \rangle$

STEP3 $\quad$ By using Lagrange multiplier :

$$\nabla T(x,y) = \lambda \nabla g(x,y)$$

$$\langle 2x+2 , 2y \rangle = \lambda \langle 2x , 8y \rangle$$

Therefore :

$2x+2 = \lambda 2x \quad$ and $\quad 2y = 8\lambda y$

$\boxed{\lambda = \frac{1}{4}}$

~~at T(-1,0)~~

$\cancel{\lambda = \frac{2x+2}{2x} = \frac{x+1}{x}}$

~~If $x=-1$ the critical point~~

$\cancel{\boxed{\lambda = 0}}$

If $\lambda = \frac{1}{4} \Rightarrow 2x+2 = \frac{1}{2}x \Rightarrow x = \frac{-4}{3}$

By substitution on $T(x,y) = x^2+4y^2 = 24$

$$y = \pm\sqrt{\frac{50}{3}}$$

The ~~max~~ extrema points are

either $T(-1,0)$

$T\left(\frac{-4}{3}, \sqrt{\frac{50}{3}}\right)$

$T\left(\frac{-4}{3}, -\sqrt{\frac{50}{3}}\right)$

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