Phys 103

Chapter 10

Rotation of a Rigid Object About a Fixed Axis

By

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LECTURE OUTLINE

Introduction

When an extended object such as a wheel rotates about its axis, the motion cannot be analyzed by treating the object as a particle because at any given time different parts of the object have different linear velocities and linear accelerations. We can, however, analyze the motion by considering an extended object to be composed of a collection of particles, each of which has its own linear velocity and linear acceleration.

In dealing with a rotating object, analysis is greatly simplified by assuming that the object is rigid. A rigid object is one that is nondeformable—that is, the relative locations of all particles of which the object is composed remain constant. All real objects are deformable to some extent; however, our rigid-object model is useful in many situations in which deformation is negligible.

10.5 Calculation of Moments of Inertia

We can evaluate the moment of inertia of an extended rigid object by imagining the object to be divided into many small volume elements, each of which has mass Δm_i.

We use the definition $I = \sum_i \Delta m_i {r_i}^2$ and take the limit of this sum as $\Delta m_i \rightarrow 0$. In this limit, the sum becomes an integral over the volume of the object:

$$I = \lim_{\Delta m_i \rightarrow 0} \sum_i {r_i}^2 \Delta m_i = \int r^2 dm \quad \because \rho = \frac{m}{V}$$

where ρ is the density of the object and V is its volume. From this equation, the mass of a small element is $\mathrm{dm} = \rho dV$ *so* $I = \int \rho\, r^2 dV$

10.5 Calculation of Moments of Inertia

$$\rho = \frac{m}{V}$$ *volumetric mass density*

$$\sigma = \rho t$$ *surface mass density*

$$\lambda = \rho A$$ *linear mass density*

10.5 Calculation of Moments of Inertia

parallel-axis theorem

The calculation of moments of inertia about an arbitrary axis can be cumbersome, however, even for a highly symmetric object. Fortunately, use of an important theorem, called the parallel-axis theorem, often simplifies the calculation. Suppose the moment of inertia about an axis through the center of mass of an object is I_{CM}. The parallel-axis theorem states that the moment of inertia about any axis parallel to and a distance D away from this axis is

$$I = I_{CM} + MD^2$$

To prove the parallel-axis theorem, uppose that an object rotates in the xy plane about the z axis, as shown in Figure 10.12, and that the coordinates of the center of mass are x_{CM}, y_{CM}. Let the mass element dm have coordinates x, y. Because this element is a distance $r = \sqrt{x^2 + y^2}$ from the z axis, the moment of inertia about the z axis is $I = \int r^2 dm = \int (x^2 + y^2) dm$

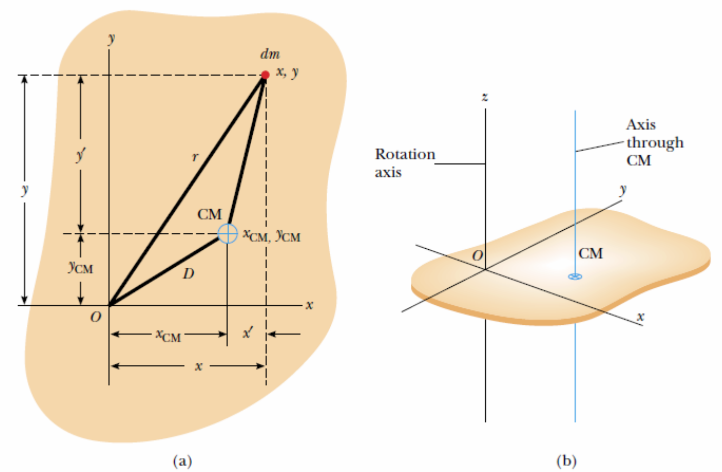


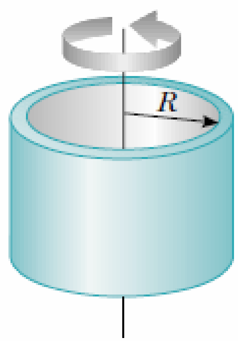
Figure 10.12 (a) The parallel-axis theorem: if the moment of inertia about an axis perpendicular to the figure through the center of mass is I_{CM}, then the moment of inertia about the z axis is $I_z = I_{CM} + MD^2$. (b) Perspective drawing showing the z axis (the axis of rotation) and the parallel axis through the CM.

10.5 Calculation of Moments of Inertia

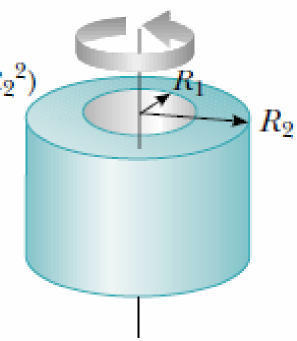
Table 10.2

Moments of Inertia of Homogeneous Rigid Objects with Different Geometries

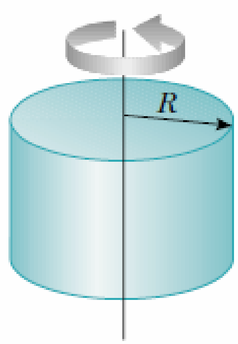
Hoop or thin cylindrical shell
$I_{CM} = MR^2$



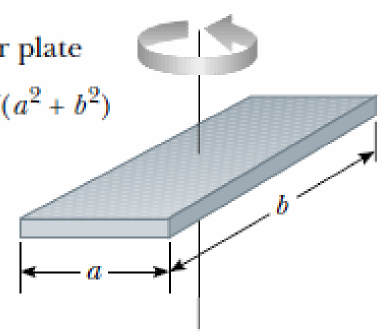
Hollow cylinder
$I_{CM} = \frac{1}{2}M(R_1^{\,2} + R_2^{\,2})$



Solid cylinder or disk
$I_{CM} = \frac{1}{2}MR^2$



Rectangular plate
$I_{CM} = \frac{1}{12}M(a^2 + b^2)$



10.5 Calculation of Moments of Inertia

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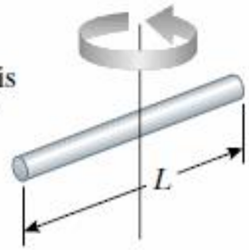
Table 10.2

Moments of Inertia of Homogeneous Rigid Objects with Different Geometries

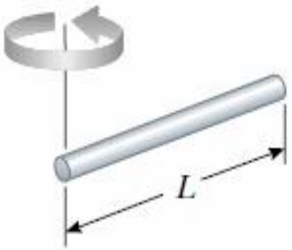
Long thin rod with rotation axis through center

$I_{CM} = \frac{1}{12} ML^2$



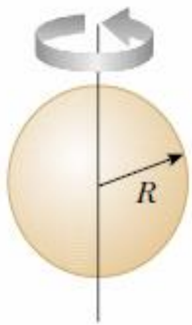
Long thin rod with rotation axis through end

$I = \frac{1}{3} ML^2$



Solid sphere

$I_{CM} = \frac{2}{5} MR^2$



Thin spherical shell

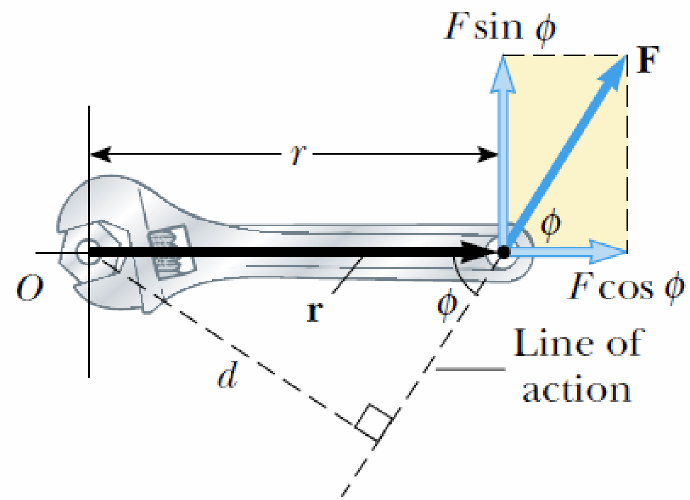
$I_{CM} = \frac{2}{3} MR^2$

10.6 Torque

The tendency of a force to rotate an object about some axis is measured by a vector quantity called torque τ.

Torque is a vector, but we will consider only its magnitude here and explore its vector nature in Chapter 11.

$$\tau = r \times F = rF\sin\phi = Fd$$



where r is the distance between the pivot point and the point of application of F and d is the perpendicular distance from the pivot point to the line of action of F. (The line of action of a force is an imaginary line extending out both ends of the vector representing the force.

10.7 Relationship Between Torque and Angular Acceleration

Consider a particle of mass m rotating in a circle of radius r under the influence of a tangential force $\mathbf{F}_t$ and a radial force $\mathbf{F}_r$, as shown in Figure 10.16. The tangential force provides a tangential acceleration $\mathbf{a}_t$, and

$$F_t = ma_t$$

The magnitude of the torque about the center of the circle due to $\mathbf{F}_t$ is

$$\tau = F_t r = (ma_t) r$$

Because the tangential acceleration is related to the angular acceleration through the relationship $a_t = r\alpha$ (see Eq. 10.11), the torque can be expressed as

$$\tau = (mr\alpha) r = (mr^2) \alpha$$

Recall from Equation 10.15 that mr^2 is the moment of inertia of the particle about the z axis passing through the origin, so that

$$\tau = I\alpha \tag{10.20}$$

That is, **the torque acting on the particle is proportional to its angular acceleration,** and the proportionality constant is the moment of inertia. Note that $\tau = I\alpha$ is the rotational analog of Newton's second law of motion, $F = ma$.

10.7 Relationship Between Torque and Angular Acceleration

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Now let us extend this discussion to a rigid object of arbitrary shape rotating about a fixed axis, as in Figure 10.17. The object can be regarded as an infinite number of mass elements dm of infinitesimal size. If we impose a Cartesian coordinate system on the object, then each mass element rotates in a circle about the origin, and each has a tangential acceleration $\mathbf{a}_t$ produced by an external tangential force $d\mathbf{F}_t$. For any given element, we know from Newton's second law that

$$dF_t = (dm)\,a_t$$

The torque $d\tau$ associated with the force $d\mathbf{F}_t$ acts about the origin and is given by

$$d\tau = r\,dF_t = a_t r\,dm$$

Because $a_t = r\alpha$, the expression for $d\tau$ becomes

$$d\tau = \alpha r^2\,dm$$

Although each mass element of the rigid object may have a different linear acceleration $\mathbf{a}_t$, they all have the *same* angular acceleration α. With this in mind, we can integrate the above expression to obtain the net torque $\sum\tau$ about O due to the external forces:

$$\sum\tau = \int \alpha r^2\,dm = \alpha\int r^2\,dm$$

where α can be taken outside the integral because it is common to all mass elements. From Equation 10.17, we know that $\int r^2\,dm$ is the moment of inertia of the object about the rotation axis through O, and so the expression for $\sum\tau$ becomes

$$\sum\tau = I\alpha \qquad (10.21)$$

10.7 Relationship Between Torque and Angular Acceleration

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Lecture Summary

If a particle moves in a circular path of radius *r through an angle θ(measured in radians),* the arc length it moves through is ***s = rθ**.*

The angular position of a rigid object is defined as the angle *θ*between a reference line attached to the object and a reference line fixed in space. The angular displacement of a particle moving in a circular path or a rigid object rotating about a fixed axis is $\Delta\theta = \theta_f - \theta_i$.

The instantaneous angular speed of a particle moving in a circular path or of a rigid object rotating about a fixed axis is: ***ω=dθ/dt***

The instantaneous angular acceleration of a particle moving in a circular path or a rotating rigid object is: ***α=dω/dt***

When a rigid object rotates about a fixed axis, every part of the object has the same angular speed and the same angular acceleration.

Thank You


ACKNOWLEDGEMENTS