PHYS 404
HANDOUT 6 – *Bessel Functions*

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1. Write the general solution of the differential equation:

$$x^2 y'' + xy' + \left(x^2 - \frac{1}{16}\right) y = 0$$

2. You are given that $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$ and $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$. (i) Find (i) $J_{3/2}(x)$ and (ii) $J_{5/2}(x)$.

3. Show that, if $\lambda \neq \mu$, then:

$$\int_0^1 x J_n(\lambda x) J_n(\mu x)\, dx = \frac{\lambda J_n(\mu) J_n'(\lambda) - \mu J_n(\lambda) J_n'(\mu)}{\mu^2 - \lambda^2}.$$

4. Show that:

$$\int_0^1 x J_n^2(\lambda x)\, dx = \frac{1}{2}\left[J_n'^2(\lambda) + \left(1 - \frac{n^2}{\lambda^2}\right) J_n^2(\lambda) \right].$$

5. A function $f(x)$ is expressed as a Bessel series: $f(x) = \sum_{n=1} b_n J_m(\alpha_{mn} x)$ with α_{mn} the nth root of J_m and $0 < x < 1$. Show that $b_n = \frac{2}{J_{m+1}^2(\alpha_{mn})} \int_0^1 x J_m(\alpha_{mn} x)\, dx$.

6. Expand the function $f(x) = 1$ in a series of the form $f(x) = \sum_{p=1} A_p J_0(\lambda_{0p} x)$, with λ_{0p} the pth root of J_0 and $0 < x < 1$.

7. A function $f(x)$ is expressed as a Bessel series: $f(x) = \sum_{n=1} a_n J_m(\alpha_{mn} x)$ with α_{mn} the nth root of J_m. Prove the Parseval's relation:

$$\int_0^1 \left[f(x) \right]^2 x\, dx = \frac{1}{2} \sum_{n=1} a_n^2 \left[J_{m+1}(\alpha_{mn}) \right]^2.$$

8. Prove that $N_{\nu+1} + N_{\nu-1} = \frac{2\nu}{x} N_\nu$.

9. Prove that $N_{\nu-1} + N_{\nu+1} = 2N_\nu'$.

10. Show that $N_{-n}(x) = (-1)^n N_n(x)$.

Calculations of integrals involving Bessel functions.

11. Calculate the integral $\int x^{n+1} J_n \, dx$.

12. Calculate the integral $\int x^3 J_0 \, dx$.

13. Discuss the calculation of integrals of the form Calculate the integral $\int x^n J_0 \, dx$. Then calculate the integral $\int x^2 J_0 \, dx$.

14. Discuss the calculation of integrals of the form Calculate the integral $\int x J_n^2 \, dx$. Then calculate the integral $\int x J_0^2 \, dx$.