PHYS 404
HANDOUT 7 – *Bessel Functions*

1. Show the following recurrence relation for the modified Bessel functions of first kind (Sch 111):

$$I_{n-1}(x) - I_{n+1}(x) = \frac{2n}{x} I_n(x)$$

2. Show the following recurrence relation for the modified Bessel functions of first kind:

$$I_{n-1}(x) + I_{n+1}(x) = 2I'_\nu(x)$$

3. Show that $I'_0(x) = I_1(x)$.

4. If ν is not integer show that (Sch 111):

$$H_\nu^{(1)}(x) = \frac{J_{-\nu}(x) - e^{-i\nu\pi} J_\nu(x)}{i\sin\nu\pi}, \quad H_\nu^{(2)}(x) = \frac{e^{i\nu\pi} J_\nu(x) - J_{-\nu}(x)}{i\sin\nu\pi}.$$

5. Show the following recurrence relation for the spherical Bessel function:

$$j_{n-1}(x) + j_{n+1}(x) = \frac{(2n+1)}{x} j_n(x)$$

(Arf. p. 628)

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6. Given that $j_0(x) = \sin x / x$ and $n_0(x) = -\cos x / x$ show that:

$$j_1(x) = \left(\sin x / x^2\right) - \left(\cos x / x\right), \quad n_1(x) = \left(\cos x / x^2\right) - \left(\sin x / x\right)$$ (Ver).

7. Show that $e^{(x/2)(t+1/t)} = \sum_{n=-\infty}^{\infty} I_n(x) t^n$, thus generating modified Bessel functions, $I_n(x)$.

8. Verify the following identities:

$$(a)\ 1 = I_0(x) + 2\sum_{n=1}^{\infty} (-1)^n I_{2n}(x)$$

$$(b)\ e^x = I_0(x) + 2\sum_{n=1}^{\infty} I_n(x)$$

9. Given that $J_n(x) = (-1)^n J_n(-x)$ verify that:

$$e^{-x} = I_0(x) + 2\sum_{n=1}^{\infty}(-1)^n I_n(x).$$

10. Verify that:

$$(a)\ \text{coshx}=I_0(x) + 2\sum_{n=1}^{\infty} I_{2n}(x)$$

$$(b)\ \text{sinhx} = 2\sum_{n=1}^{\infty} I_{2n-1}(x)$$

11. From $K_\nu(x) = \frac{\pi}{2}\frac{I_{-\nu}(x) - I_\nu(x)}{\sin\nu\pi}$ and show that: $K_{-\nu}(x) = K_\nu(x)$.

12. Show that $K_\nu(x)$ satisfies following recurrence relation:

$$K_{\nu-1}(x) - K_{\nu+1}(x) = -\frac{2\nu}{x}K_\nu(x)$$

$$K_{\nu-1}(x) + K_{\nu+1}(x) = -2K_\nu^{'}(x)$$

13. Derive the integral representation $I_n(x) = \frac{1}{\pi}\int_0^\pi e^{x\cos\theta}\cos(n\theta)\,d\theta$. (Hint: start from the corresponding integral representation of $J_n(x)$.

14. Prove that

$$\sum_{n=1}^{\infty}(\alpha_{mn})^{-2} = \frac{1}{4(m+1)}.$$

Hint: Expand x^m in a Bessel series and apply the Parseval relation.

15. Show that:

$$\int_{-\infty}^{\infty}\left[j_n(x)\right]^2 dx = \frac{\pi}{2n+1},$$

(Arf. p. 631)