PHYSICS 404
1st HOMEWORK - SOLUTIONS
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Hand in: Tuesday 17th of October 2017

1. Show that for Dirac delta function we have $\delta(1+x)=\sum_{n=0}^{\infty}(-1)^n \frac{2n+1}{2} P_n(x)$

$$\delta(1+x)=\sum_{n=0}^{\infty} A_n P_n(x) \Rightarrow A_n=\frac{(2n+1)}{2}\int_{-1}^{+1}\delta(1+x)P_n(x)dx \Rightarrow$$
$$A_n=\frac{(2n+1)}{2}P_n(-1)=\frac{(2n+1)}{2}(-1)^n$$

2. Show that for the Legendre polynomials we have:
$$P'_{n+1}(x)=(n+1)P_n(x)+xP'_n(x).$$

Hint: use the recurence relations: $P'_{n+1}(x)+P'_{n-1}(x)=2xP'_n(x)+P_n(x)$ and $(n+1)P_{n+1}(x)=(2n+1)xP_n(x)-nP_{n-1}(x)$.

We have

$$(n+1)P_{n+1}(x)=(2n+1)xP_n(x)-nP_{n-1}(x)\Rightarrow$$
$$(n+1)P'_{n+1}(x)=(2n+1)P_n(x)+(2n+1)xP'_n(x)-nP'_{n-1}(x) \qquad (1)$$

Also

$$P'_{n+1}(x)+P'_{n-1}(x)=2xP'_n(x)+P_n(x)\Rightarrow$$
$$-nP'_{n+1}(x)-nP'_{n-1}(x)=-2nxP'_n(x)-nP_n(x) \qquad (2)$$

Add now (1)+(2) and you get

$$(n+1)P'_{n+1}(x)-nP'_{n+1}(x)-nP'_{n-1}(x)=(2n+1)P_n(x)+(2n+1)xP'_n(x)-nP'_{n-1}(x)-2nxP'_n(x)-nP_n(x)$$

$$(n+1)P'_{n+1}(x)-nP'_{n+1}(x)-nP'_{n-1}(x)+nP'_{n-1}(x)=(2n+1)P_n(x)+(2n+1)xP'_n(x)-2nxP'_n(x)-nP_n(x)$$

$$P'_{n+1}(x) + P'_{n-1}(x) = 2xP'_n(x) + P_n(x)$$

3. Find the associated Legendre functions $P_2^1(x)$ and $P_3^1(x)$ starting from the Legendre polynomials $P_2\,(x)$ and $P_3\,(x)$.

$$P_2^1(x) = \left(1-x^2\right)^{1/2} \frac{d}{dx} P_2(x) \underset{P_2(x)=\frac{1}{2}\left(3x^2-1\right)}{\Rightarrow} P_2^1(x) = \left(1-x^2\right)^{1/2} \frac{d}{dx}\left[\frac{1}{2}\left(3x^2-1\right)\right] \Rightarrow$$

$$P_2^1(x) = \frac{1}{2}\left(1-x^2\right)^{1/2} 6x \Rightarrow P_2^1(x) = 3x\left(1-x^2\right)^{1/2}$$

Similarly we have:

$$P_3^1(x) = \left(1-x^2\right)^{1/2} \frac{d}{dx} P_3(x) \underset{P_3(x)=\frac{1}{2}\left(5x^3-3x\right)}{\Rightarrow} P_3^1(x) = \left(1-x^2\right)^{1/2} \frac{d}{dx}\left[\frac{1}{2}\left(5x^3-3x\right)\right] \Rightarrow$$

$$P_3^1(x) = \frac{1}{2}\left(1-x^2\right)^{1/2}\left(15x^2-3\right) \Rightarrow P_3^1(x) = \frac{3}{2}\left(1-x^2\right)^{1/2}\left(5x^2-1\right)$$

4. Using your answers from question 3, show that the functions $P_2^1(x)$ and $P_3^1(x)$ are orthogonal i.e. $\int_{-1}^{1} P_2^1(x) P_3^1(x)\,dx = 0$.

$$\int_{-1}^{1} P_2^1(x) P_3^1(x)\,dx = \int_{-1}^{1} 3x(1-x^2)^{1/2} \frac{3}{2}(1-x^2)^{1/2}\left(5x^2-1\right)dx =$$

$$\frac{9}{2}\int_{-1}^{1} x(1-x^2)\left(5x^2-1\right)dx = \frac{9}{2}\int_{-1}^{1}\left(-5x^5+6x^3-x\right)dx =$$

$$\frac{9}{2}\left[-5\int_{-1}^{1} x^5 dx + 6\int_{-1}^{1} x^3 dx - \int_{-1}^{1} x\,dx\right] = \frac{9}{2}\left[-5\frac{x^6}{6}\Bigg|_{-1}^{+1} + 6\frac{x^4}{4}\Bigg|_{-1}^{+1} - \frac{x^2}{2}\Bigg|_{-1}^{+1}\right] =$$

$$\frac{9}{2}\left[-5\frac{x^6}{6}\Bigg|_{-1}^{+1} + 6\frac{x^4}{4}\Bigg|_{-1}^{+1} - \frac{x^2}{2}\Bigg|_{-1}^{+1}\right] = \frac{9}{2}\left[\frac{-5}{6}\left(1^6-(-1)^6\right) + \frac{6}{4}\left(1^4-(-1)^4\right) - \frac{1}{6}\left(1^2-(-1)^2\right)\right] = 0$$