PHYSICS 404
FALL 2019
1st HOMEWORK
Dr. V. Lempesis

Hand in: Tuesday 24th of September 2019

1. Use the Rodrigues formula and find the Legendre polynomial $P_4(x)$.

Vasileios Lembessis 24/9/2019 13:55
Comment [1]: Only solutions using Rodriguez formula are accepted as correct.

Solution

The Rodriguez formula is given by:

$$P_n(x) = \frac{1}{2^n n!}\left(\frac{d}{dx}\right)^n \left(x^2-1\right)^n .$$

Thus

$$P_4(x) = \frac{1}{2^4 4!}\left(\frac{d}{dx}\right)^4 \left(x^2-1\right)^4 = \frac{1}{384}\left(\frac{d}{dx}\right)^4 \left(x^8-4x^6+6x^4-4x^2+1\right) =$$

$$\frac{1}{384}\left(\frac{d}{dx}\right)^3 \left(8x^7-24x^5+24x^3-8x\right) = \frac{1}{384}\left(\frac{d}{dx}\right)^2 \left(56x^6-120x^4+72x^2-8\right) =$$

$$\frac{1}{384}\left(\frac{d}{dx}\right)\left(336x^5-480x^3+144x\right) = \frac{1}{384}\left(1680x^4-1440x^2+144\right) =$$

$$\frac{1680}{384}x^4 - \frac{1440}{384}x^2 + \frac{144}{384} = \frac{105}{24}x^4 - \frac{90}{24}x^2 + \frac{12}{32} = \frac{35}{8}x^4 - \frac{30}{8}x^2 + \frac{3}{8} =$$

$$\frac{1}{8}\left(35x^4-30x^2+3\right)$$

2. Show that:

$$\left(1-x^2\right)P_n'(x) = nP_{n-1}(x) - nxP_n(x) .$$

Hint: use the recurence relations: $P_{n+1}'(x) = (n+1)P_n(x) + xP_n'(x)$ and $P_{n-1}'(x) = -nP_n(x) + xP_n'(x)$.

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Comment [2]: Both relations have to be used in your answer.

Solution

$$P_{n+1}'(x) = (n+1)P_n(x) + xP_n'(x) \underset{n\to n-1}{\Rightarrow} P_n'(x) = nP_{n-1}(x) + xP_{n-1}'(x) \qquad (1)$$

Also

$$P_{n-1}^{'}(x) = -nP_n(x) + xP_n^{'}(x) \Rightarrow xP_{n-1}^{'}(x) = -nxP_n(x) + x^2 P_n^{'}(x) \qquad (2)$$

Adding (1) and (2) two we have

$$P_n^{'}(x) + xP_{n-1}^{'}(x) = nP_{n-1}(x) + xP_{n-1}^{'}(x) + -nxP_n(x) + x^2 P_n^{'}(x)$$

$$(1-x^2)P_n^{'}(x) = nP_{n-1}(x) - nxP_n(x)$$

3. Calculate the integral $\int_{-1}^{1}(x^2-1)P_n^{'}(x)P_{n+1}(x)\,dx$. (Hint: use the first and last recurrence relations in slide 15 of Lecture 1)

Solution:

From the last recurrence relation we have:

$$\left(1-x^2\right)P_n^{'}(x) = nP_{n-1}(x) - nxP_n(x)$$

Thus

$$\int_{-1}^{1}(x^2-1)P_n^{'}(x)P_{n+1}(x)\,dx = \int_{-1}^{1}\left[nP_{n-1}(x) - nxP_n(x)\right]P_{n+1}(x)\,dx =$$
$$n\int_{-1}^{1} P_{n-1}(x)P_{n+1}(x)dx - n\int_{-1}^{1} xP_n(x)P_{n+1}(x)dx$$

Since $P_{n-1}(x)$ and $P_{n+1}(x)$ are orthogonal the first integral is zero, thus

$$\int_{-1}^{1}(x^2-1)P_n^{'}(x)P_{n+1}(x)\,dx = -n\int_{-1}^{1} xP_n(x)P_{n+1}(x)dx$$

From the first recurrence relation in slide 15 we have

$$\left(2n+1\right)xP_n(x) = \left(n+1\right)P_{n+1}(x) + nP_{n-1}(x) \Rightarrow xP_n(x) = \frac{1}{\left(2n+1\right)}\left[\left(n+1\right)P_{n+1}(x) + nP_{n-1}(x)\right]$$

$$\Rightarrow xP_n(x) = \frac{\left(n+1\right)}{\left(2n+1\right)}P_{n+1}(x) + \frac{n}{\left(2n+1\right)}P_{n-1}(x)$$

So we have

$$\int_{-1}^{1}(x^2-1)P_n'(x)P_{n+1}(x)\,dx=-n\int_{-1}^{1}xP_n(x)P_{n+1}(x)dx=$$

$$-n\int_{-1}^{1}\left[\frac{(n+1)}{(2n+1)}P_{n+1}(x)+\frac{n}{(2n+1)}P_{n-1}(x)\right]P_{n+1}(x)\,dx=$$

$$-\frac{n(n+1)}{(2n+1)}\int_{-1}^{1}\left[P_{n+1}(x)\right]^2dx-\frac{n^2}{(2n+1)}\int_{-1}^{1}P_{n-1}(x)P_{n+1}(x)\,dx=$$

$$-\frac{n(n+1)}{(2n+1)}\int_{-1}^{1}\left[P_{n+1}(x)\right]^2dx=-\frac{n(n+1)}{(2n+1)}\frac{2}{2(n+1)+1}=-\frac{2n(n+1)}{(2n+3)(2n+1)}$$

4. Find the general solution of the differential equation $(1-x^2)y''-2xy'+6y=0$

Solution:

The equation can be written as

$$(1-x^2)y''-2xy'+2\cdot 3y=0\Rightarrow(1-x^2)y''+\left(1-x^2\right)'y'+2\cdot(2+1)y=0\Rightarrow$$

$$\left[(1-x^2)y'\right]'+2\cdot(2+1)y=0$$

This is the Legendre diff. equation for n = 2, and thus it has the general solution

$y(x)=AP_2(x)+BQ_2(x)$.

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Comment [3]: Do not forget that we have also Q in the general solution!