The Poisson Distribution

Recall: The Taylor series expansion of the function $f(x)=e^{x}$

$$e^{x}=\sum_{i=0}^{\infty}\frac{x^{i}}{i!}=1+x+\frac{x^{2}}{2!}+\frac{x^{3}}{3!}+\dots$$

Let X be a r.v. with pdf given by

$$p_X(x)=e^{-\lambda}\frac{\lambda^{x}}{x!} \quad \text{for } x\in\mathbb{N}_0=\{0,1,2,3,\dots\}$$

then X is said to follow a Poisson distribution with parameter λ. Notationally $X\sim Pois(\lambda)$

Facts: Suppose $X\sim Pois(\lambda)$

i) $$\sum_{x=0}^{\infty}p_X(x)=\sum_{x=0}^{\infty}e^{-\lambda}\frac{\lambda^{x}}{x!}=e^{-\lambda}\underbrace{\sum_{x=0}^{\infty}\frac{\lambda^{x}}{x!}}_{=e^{\lambda}}=e^{-\lambda}e^{\lambda}=1$$

ii) $$E_X[X]=\sum_{x=0}^{\infty}xe^{-\lambda}\frac{\lambda^{x}}{x!}=\sum_{x=1}^{\infty}xe^{-\lambda}\frac{\lambda^{x}}{x!}=\lambda\sum_{x=1}^{\infty}e^{-\lambda}\frac{\lambda^{x-1}}{(x-1)!} \quad \text{reindex } u=x-1$$

$$=\lambda\underbrace{\sum_{u=0}^{\infty}e^{-\lambda}\frac{\lambda^{u}}{u!}}_{=1}=\lambda$$

iii) $Var_X[X]=\lambda$

Pf.

Consider $$E[X(X-1)]=\sum_{x=0}^{\infty}x(x-1)e^{-\lambda}\frac{\lambda^{x}}{x!}=\sum_{x=2}^{\infty}x(x-1)e^{-\lambda}\frac{\lambda^{x}}{x!}$$

$$=\lambda^{2}\sum_{x=2}^{\infty}e^{-\lambda}\frac{\lambda^{x-2}}{(x-2)!}\underset{\substack{\text{reindex}\\ u=x-2}}{=}\lambda^{2}\underbrace{\sum_{u=0}^{\infty}e^{-\lambda}\frac{\lambda^{u}}{u!}}_{=1}=\lambda^{2}$$

So $E_X[X(X-1)]=E_X[X^{2}]-E_X[X]=E_X[X^{2}]-\lambda=\lambda^{2}$

Thus $E_x[X^2] = \lambda^2 + \lambda$. Consequently

$Var_x[X] = E_x[X^2] - E_x^2[X] = \lambda^2 + \lambda - \lambda^2 = \lambda$. QED

iv) $m_x(t) = e^{\lambda(e^t - 1)}$

Pf.

$$m_x(t) = E_x[e^{tx}] = \sum_{x=0}^{\infty} e^{tx} e^{-\lambda} \frac{\lambda^x}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{(e^t \lambda)^x}{x!}$$

$$= e^{-\lambda} e^{e^t \lambda} = e^{\lambda(e^t - 1)} \quad \text{QED}$$

<u>Result</u>: Suppose $X_1, X_2, \ldots, X_n$ be independent with $X_i \sim Pois(\lambda_i)$.

define

$$S = \sum_{i=1}^{n} X_i$$

Then it follows $S \sim Pois\left(\sum_{i=1}^{n} \lambda_i\right)$

Pf.

Find $m_S(t)$. $m_S(t) = \prod_{i=1}^{n} m_{X_i}(t) = \prod_{i=1}^{n} e^{\lambda_i(e^t - 1)} = e^{\sum_{i=1}^{n} \lambda_i (e^t - 1)}$

$$= e^{\left(\sum_{i=1}^{n} \lambda_i\right)(e^t - 1)}$$

Above is the mgf of a Poisson r.v. with parameter $\sum_{i=1}^{n} \lambda_i$. QED

Ex 1) Suppose $X \sim Pois(\lambda)$ and $Y \sim Pois(\eta)$ with X and Y independent. Find the distribution of X conditional on the event $X + Y = t$.

Note: Obviously $X \in \{0, 1, \ldots t\}$. Apply Bayes Rule

$$P_r(X = x \mid X + Y = t) = \frac{P_r((X = x) \cap (X + Y = t))}{P_r(X + Y = t)}$$

$$= \frac{P_r((X = x) \cap (Y = t - x))}{P_r(X + Y = t)}.$$

Use the fact $X + Y \sim Pois(\lambda + \eta)$ & independence of X and Y

$$= \frac{Pr(X=x)\,P(Y=t-x)}{Pr(X+Y=t)} = \left(e^{-\lambda}\frac{\lambda^x}{x!}\right)\left(e^{-\eta}\frac{\eta^{t-x}}{(t-x)!}\right) \Big/ \left(e^{-(\lambda+\eta)}\frac{(\lambda+\eta)^t}{t!}\right)$$

$$= \frac{t!}{x!(t-x)!}\frac{\lambda^x \eta^{t-x}}{\underbrace{(\lambda+\eta)^{t-x+x}}_{\text{trick}}} = {}_tC_x \underbrace{\left(\frac{\lambda}{\lambda+\eta}\right)^x}_{0<\cdot<1} \underbrace{\left(\frac{\eta}{\lambda+\eta}\right)^{t-x}}_{0<\cdot<1} \quad \text{let } p = \frac{\lambda}{\lambda+\eta} \text{ so } 1-p = \frac{\eta}{\lambda+\eta}$$

$$= {}_tC_x\, p^x (1-p)^{t-x}$$

So $X \mid (X+Y=t) \sim Bin\left(t, \frac{\lambda}{\lambda+\eta}\right)$.

Ex 2) Let $X_0 \sim Gamma(K, \lambda)$, where $K \in \mathbb{N}$, it follows

$$Pr(X_0 \le x) = \frac{\lambda^K}{\Gamma(K)}\int_0^x t^{K-1}e^{-t\lambda}\,dt$$

integrate by parts let $u = t^{K-1}$ $dv = e^{-t\lambda}$

$du = (K-1)t^{K-2}dt$ $v = \frac{-1}{\lambda}e^{-t\lambda}$

$$= \frac{\lambda^K}{(K-1)!}\left\{ \frac{-1}{\lambda}t^{K-1}e^{-t\lambda}\Big|_0^x - \int_0^x \left(\frac{-1}{\lambda}\right)(K-1)t^{K-2}e^{-t\lambda}\,dt \right\}$$

$$= \frac{\lambda^{K-1}}{(K-2)!}\int_0^x t^{K-2}e^{-t\lambda}\,dt - \frac{\lambda^{K-1}}{(K-1)!}x^{K-1}e^{-\lambda x}.$$

Let $Y \sim Pois(x\lambda)$

$X_1 \sim Gamma(K-1, \lambda)$

$$= Pr(X_1 \le x) - P(Y = K-1)$$

For $Pr(X_1 \le x)$, integrate by parts, let $u = t^{K-1}$, $dv = e^{-t\lambda}$ to show

$Pr(X_1 \le x) = Pr(X_2 \le x) - P(Y = K-2)$ where $X_2 \sim Gamma(K-2, \lambda)$

Repeat

$$Pr(X_0 \le x) = Pr(X_{K-1} \le x) - \sum_{i=1}^{K-1} P(Y = K-i) \quad \text{where } X_{K-1} \sim Gamma(1, \lambda) = Exp(\lambda)$$

$$= \lambda\int_0^x e^{-t\lambda} - \sum_{i=1}^{K-1} Pr(Y=K-i) = \lambda\left(\frac{-1}{\lambda}e^{-t\lambda}\Big|_0^x\right) - \sum_{i=1}^{K-1} Pr(Y=K-i)$$

$$= 1 - \underbrace{e^{-\lambda x}}_{Pr(Y=0)} - \sum_{i=1}^{K-1} Pr(Y=K-i) = 1 - Pr(Y=0) - Pr(Y=K-1) - \cdots - Pr(Y=1)$$

$$= 1 - \sum_{i=0}^{K-1} Pr(Y=i) = Pr(Y \ge K)$$